Noncooperative oligopoly equilibrium in markets with hierarchical competition

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Noncooperative Oligopoly Equilibrium in Markets with Hierarchical Competition

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This paper deals with the existence of a non-cooperative sequential equilibrium in interrelated markets with heterogeneous atomic traders. Since this model features a rich set of strategic interactions, there are two kinds of problems associated with the existence of equilibrium. First, existence and uniqueness of followers' strategies are not guaranteed. Second, the no-trade equilibrium is always an equilibrium outcome. To overcome these two difficulties we consider a differentiable approach. We show that the set of equations which determines the strategies of followers is a variety with the required dimension, i.e. the vector mapping which defines this set is a local $\mathcal{C}^2$-diffeomorphism. The continuous differentiability of followers' strategies is critical for the existence of an interior equilibrium. Unlike the simultaneous move games, exchange can take place in one subgame while autarky can hold in another subgame, in which case only leaders (followers) make trade. Some examples buttress the approach and discuss the assumptions made on the primitives.

Key Words: Pure strategies, diffeomorphisms, Stackelberg-Nash equilibrium
Subject Classification: C72, D51

1. INTRODUCTION

The coordination of private activities in decentralized markets is a core issue. This issue found a definitive solution in general competitive equilibrium models (Arrow and Debreu 1954), and in strategic market game models with Cournot competition (Dubey 1994). But the existence of equilibrium remains an open problem when strategic competition in interrelated markets is hierarchical, i.e. when the competition is of the Stackelberg type. The starting point of this paper is to consider hierarchical competition between strategic buyers and sellers. Therefore, we propose to build a Stackelberg game in which decentralized exchange embodies only strategic traders. Our main objective is to prove the existence of a Stackelberg-Nash equilibrium with trade within the research program for strategic market games founded by Shubik (1973), and Shapley and Shubik (1977).

1.1. Motivations

The motivations are twofold. First, this contribution studies hierarchical optimization problems into a general equilibrium setting. These problems have been studied in multiple leader-follower games in which individuals who belong to one industry decide in a sequential way (Sherali 1984). Here we consider a two-sector

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multiple leader-follower game in the framework of an exchange economy.[2] There are two types of commodities. The agents are categorized according to which of the two types of commodities they trade. Traders of type X (resp. Y) have an initial endowment of commodity X (resp. Y) that they wish to trade for amounts of commodity Y (resp. X).[3] Within each type, traders may be either leaders or followers. Leaders decide in the first stage, anticipating the followers' reactions, and followers decide in the second stage. There is a market-clearing relative price which aggregates the supplies of all traders over both stages and allocates the amounts traded to each trader. Each trader maximizes an utility function which depends on the remaining quantity of her own commodity and the quantity of the other commodity that she obtains after trade. Therefore, there are two multiple leader-follower industries which are connected through trade. Possible applications should include product differentiation in two-sided markets (water markets, mobile telephony markets), communication networks, spatial economics with locational interdependencies, and international trade with resource specialization.

Why is it important to study the multiple leader-follower game in the context of interrelated markets? In our setting all agents behave strategically: the demand side as well as the supply side reflect strategic behavior. Here the traders make rational decisions as buyers and sellers. Indeed, the preferences of traders provide some micro-foundations for the market demand, and then for the price function. By contrast, in the multiple leader-follower games market demand is given insofar as it is assumed that buyers behave competitively as price takers. In addition, the supply is also the result of a decision: as suppliers the agents bring to the market the difference between the amount they hold and the amount they decide to consume. The price function is endogenous: there is a market price mechanism that makes strategic supplies mutually compatible. Indeed, the supply by the traders of type X (resp. type Y) matches the supply by the traders of type Y (resp. X).

Second, the main objective is to study the existence of a noncooperative equilibrium with trade in a two-sector model with a finite number of leaders and followers. To this end, our model extends the Cournot-Nash bilateral oligopoly model introduced by Gabszewicz and Michel (1997), and explored by Bloch and Ferrer (2001), Dickson and Hartley (2008), and Amir and Bloch (2009) among others.[4] In this model, there is a finite number of strategic buyers and sellers. Each trader has corner endowment but wants to consume both commodities. There is a market price which aggregates the strategic supplies of all traders and allocates the amounts traded to each market participant. The Cournot-Nash equilibrium (CNE) is the equilibrium concept. This leads us to define, within this framework, another strategic equilibrium concept, namely the Stackelberg-Nash equilibrium (SNE), as the equilibrium outcome of this extended multiple leader-follower game (Section 3). The SNE may be viewed as the solution concept of a generalized Stackelberg game: two Cournot subgames are embedded in a two-stage sequential game. Therefore, the existence of a non-cooperative equilibrium is quite challenging.

[2] The modeling of production activities in interrelated imperfectly competitive markets raise some difficulties (Gabszewicz and Vial 1972, Dierker and Grodal 1999). It turns out that the exchange model is a natural starting point to consider new issues in interrelated markets.

[3] If Y is viewed as commodity money, i.e. a numeraire, then quantities of Y (resp. X) are bids (resp. offers) and the corresponding agents are buyers (resp. sellers).

[4] The bilateral oligopoly model is a two-good version of the strategic market games introduced by Shubik (1973), Shapley and Shubik (1977). See Giraud (2003), and Dickson and Tonin (2021) for surveys.

This paper provides new insights on the study of existence of Stackelberg-Nash equilibria. First, it gives non-cooperative foundations for bilateral exchange. Indeed, the fact that individual demand behavior is strategic is of significant importance for trade. It turns out that the preferences of leaders and followers may lead them to choose not to trade, and the outcome of the game may be the no trade equilibrium. This leads us to consider the possibility of autarkic equilibria in the multiple leader-follower game. Second, Stackelberg competition also provides new insights on the study of optimal behavior in bilateral oligopolies. To be specific, by introducing sequential heterogeneous behavior, the characterization of the strategic equilibrium, i.e. the study of the optimal behavior in each subgame, brings into light some hierarchical strategic interaction for which leaders anticipate the reactions of each follower. Indeed, the followers' strategies tremendously matter for the exploration of the mechanisms at work in the hierarchical interactions of sequential games involving several heterogeneous agents who behave strategically.

1.2. Our contribution

The main contribution of the paper is a theorem which proves the existence of a non-cooperative sequential equilibrium with trade in a strategic market game with a finite set of atoms. To prove the existence of a SNE with trade, we consider a slight perturbation of the two-stage bilateral oligopoly model. Then, we show that the resulting equilibria in the perturbed game exist, and then that equilibria to the original game exist. The proof of our theorem requires five steps to which five lemmas correspond. To the best of our knowledge, no general existence result has been yet obtained about Stackelberg-Nash equilibria in interrelated markets.[5]

There are two main problems involved with the existence of a SNE with trade. The first problem, which is linked to the structure of the game, concerns the followers' strategies (Julien 2017). In the basic one leader-one follower game, under mild technical assumptions, the follower's strategy coincides with his best response: it is determined, given any strategy profile of the leader, as the solution to the maximization of the follower's payoff. But with at least two followers, any follower's best reply consists of a mapping which depends on two kinds of arguments: the strategies of leaders and the strategies of all *other* followers. To determine the followers' strategies, the best responses must be mutually consistent. In case they are not, which is a possible outcome in decentralized systems without central coordination, then neither the strategies nor the price function that maps leaders' strategies into a price could exist. An example in Section 5 illustrates this possibility.

It is worth noting that our notion of consistency differs from the notion of price consistency in Leiffer and Munson (2010) that results in a square nonlinear complementarity problem. In their approach, the leaders' problems constraints can violate the Mangasarian-Fromovitz qualification constraint as there can be an infinite numbers of multipliers, so they assume that the vector of unique shadow prices is set by an independent entity (leaders' strategy sets are independent of followers' decisions). In our approach, the consistency is based on the mutual compatibility between the followers' best responses, which makes it possible to define the followers' strategies. Our notion of consistency also differs from that of Kurkarni and Shanbhag (2014) who use a shared-constraint approach which does not require uniqueness of best responses. Here the best responses are not multi-valued.

[5] Some Stackelberg equilibrium concepts for the multiple leader-follower game are defined and computed in finite exchange economies (Julien 2013).

To circumvent the first problem, we develop a new approach based on differentiability. The characterization of the strategic equilibrium, i.e. the optimal behavior in each subgame, brings to light a consistency condition. This condition is specific to sequential multiple decision settings, and relates to the internal consistency of the system of equations which determines the followers' strategies. Indeed, by using the collection of followers' best replies, we define a vector function from which we set up a system of equations that defines implicitly the strategies. Then, we provide a nondegeneracy condition on the determinant of the Jacobian matrix associated with this system of equations. This consistency condition is sufficient (Lemma 1) for the existence and uniqueness of continuously differentiable strategies. Our approach is based on local $\mathcal{C}^2$-diffeomorphisms, i.e. on twice continuously differentiable invertible mappings in the neighborhood of a point. To this end, we resort to one version of the Implicit Function Theorem for differentiable mappings in Banach space (Raeburn 1979; Dontchev and Rockafellar 2014). If the Jacobian of the vector function is an invertible mapping, to be specific a $\mathcal{C}^1$-diffeomorphism, then there exists a unique fixed point which consists of twice continuously differentiable strategies. Then, the price vector function that maps leaders' strategies into a price vector as well as the reduced form payoffs of leaders exist.

The second problem is related to the possibility of autarky in decentralized markets. It is well known that the trivial equilibrium is always a Nash equilibrium in strategic market games with simultaneous moves (Cordella and Gabszewicz 1998; Busetto and Codognato 2006). Thus, we wonder whether autarky is a plausible outcome in the multiple leader-follower game with heterogeneous traders. Even if there exists a SNE in the perturbed game (Lemma 2), it seems plausible to conjecture that the no trade equilibrium is a possible outcome for the entire sequential game, in which case neither leaders nor followers participate in exchange. In this respect, our contribution provides new insights on the study of optimal behavior in bilateral oligopolies. Indeed, exchange can take place in one subgame with autarky in the other subgame, in which case only the leaders or only the followers make trade. An example in Section 4 provides an illustration. This salient feature is precluded in bilateral oligopoly with simultaneous moves.

To circumvent the second problem, we consider a slight perturbation of the game, and we adapt to our setting the Uniform Monotonicity Lemma of Dubey and Shubik (1978). It is worth noting that, unlike the existence of uniform bounds on relative price in simultaneous move games, the existence of such uniform bounds is more difficulty to handle with as it must hold in each perturbed subgame of the sequential game. Indeed, we have to show that the market price is bounded in each stage of the perturbed game (Lemma 3). More specifically, we take into account that, in the perturbed subgame between leaders, the rational beliefs of leaders about the followers' reactions matter. Then, we can show that the existence of a SNE with trade in the perturbed game (Lemma 4), and finally that the SNE is an equilibrium point of the game (Lemma 5), i.e. a non trivial subgame perfect Nash equilibrium which is robust to slight perturbation of the game.

It turns out that both problems, namely the existence of strategies and the possibility of autarky are closely related. Indeed, the existence of a SNE with trade for the entire game depends on the mutual consistency of the best replies in the subgame between followers. Under this consistent condition, the reduced form payoffs of leaders exist, and the existence of pure strategy subgame perfect Nash equilibria (with trade) in the finite extensive form game can be studied.

1.3. Related literature

From a methodological viewpoint, our model crosses two types of literature on noncooperative equilibria: the multiple leader-follower games and the bilateral oligopoly models.

Existence has been explored in the multiple leader-follower model. Sherali (1984) shows existence and uniqueness with identical convex costs for leaders, and states some results relating to the properties of the aggregate best response under the assumptions of linear demand with either linear or quadratic costs (Ehrenmann, 2004). The determination of the convex best response stems from a family of optimization programs for the followers based on a price function which is affected by the supply of the leaders. Neverthess, the conditions under which the followers' decisions are mutually consistent are not studied. De Miguel and Xu (2009) include uncertainty with stochastic market demand. Unlike Sherali (1984) they allow costs to differ across leaders. But, to show that the expected profit of any leader is concave, they assume that the aggregate best response of the followers is convex. As this assumption does not always hold, these authors must resort to a linear demand. Su (2007) studies existence of an equilibrium in the two-period forward market model where each player solves a nonconvex program with equilibrium constraints under the assumptions of linear demand and constant marginal costs.

Fukushima and Pang (2005), Yu and Wang (2008), Hu and Fukushima (2011), and Jia et al. (2015) prove the existence of an equilibrium point with two leaders and several followers without specifying demand and costs. Aussel and Dutta (2008) prove existence of a Nash equilibrium by using the quasivariational inequality approach, but without considering market demand. Kurkarni and Shanbhag (2015) show that when the leaders' objectives admit a quasi-potential function, the global and local minimizers of the leaders' optimization problems are global and local equilibria of the game. The novelty of our approach with respect to the multiple leader-follower game is twofold. First, by considering all agents behave strategically, we study the conditions under which followers' strategies exist. Second, we consider a framework in which demand behavior is micro-founded.

To this end, we turn to the class of non-cooperative bilateral oligopoly models with a finite number of traders introduced by Gabszewicz and Michel (1997). Such models have been widely studied under the assumption of Cournot competition. Bloch and Ghosal (1997) study existence and uniqueness of Cournot-Nash equilibria with trade under the assumption that traders have the same utility function. Bloch and Ferrer (2001) show the existence of Cournot-Nash equilibria with trade by allowing heterogeneity in preferences represented by strictly convave utility functions. By using an aggregate game approach for which the payoff of each trader depends on the strategies of all other traders only through aggregate offers and bids (the same for all traders), Dickson and Hartley (2008) define strategic versions of Marshallian supply and demand curves, and they prove the existence and uniqueness of Cournot-Nash equilibria with trade assuming only that the preferences of traders are normal in both goods and satisfy a weak version of gross substitutes. Amir and Bloch (2009) focus on comparative statics, and only impose symmetry on each side of the market, allowing buyers to have different preferences from sellers. They show that gross substitutes imply uniqueness of equilibrium. More recently, Busetto et al. (2020) show the existence of a Cournot-Nash equilibrium for the mixed bilateral oligopoly version of the Shapley window model, i.e. with atoms and an atomless part. They notably impose that there is a coalition of traders in the atomless part with differentiable and additively separable utility functions which have infinite partial derivatives along the boundary of the consumption set.

To the best of our knowledge, no general existence proof of a Stackelberg equilibrium has been yet obtained in bilateral oligopoly with a finite number of traders. Indeed, our model differs from the previous ones insofar as the existence of a sequential equilibrium requires us to specify the optimal strategic behavior of heterogeneous traders at each stage of the game. Groh (1999) studies an example of bilateral oligopoly with leaders as sellers and followers as buyers. The existence of a sequential equilibrium with trade relies on three restrictions: the utility function is quadratic; each side of the market embodies only leaders or followers (a leader is only a seller and a follower only a buyer); and, traders are identical within each side of the bilateral market. Our contribution goes beyond these three shortcomings: we consider a general class of smooth utility functions, and heterogeneous leaders and followers compete within each side and between both sides of the market.

1.4. Content

The paper is organized as follows. In section 2, we describe the model, and we define the Stackelberg-Nash equilibrium. Section 3 is devoted to the existence of a Stackelberg-Nash equilibrium with trade. Section 4 provides some examples to discuss the assumptions, buttress the working of our approach, and put forward the main differences with the corresponding Cournot-Nash games. In section 5 we conclude. An appendix collects some proofs.

1.5. Notations

Consider the following notational convention. Vectors are in bold and capital letters denote either sets or summations. Let $\mathbf{z} \in \mathbb{R}^n_+$. Then, $\mathbf{z} \geq \mathbf{0}$ means $z_i \geqslant 0$, $i = 1, ..., n$; $\mathbf{z} > \mathbf{0}$ means there is some i such that $z_i > 0$, with $\mathbf{z} \neq \mathbf{0}$, and $\mathbf{z} >> \mathbf{0}$ means $z_i > 0$ for all i, $i = 1, ..., n$. Let $z_i \geqslant 0$ be an action. An action profile is given by $\mathbf{z} = (z_1, ..., z_i, ..., z_n)$, with $\mathbf{z} \geq \mathbf{0}$. In addition, let $\mathbf{z}_{-i} \triangleq (z_1, ..., z_{i-1}, z_{i+1}, ..., z_n)$. We sometimes set $Z \equiv \sum\limits_{i=1}^{n} z_i$, with $Z_{-i} \triangleq \sum\limits_{-i,-i \neq i} z_{-i} = Z - z_i$. The Cartesian product of sets A_i is denoted by $\prod\limits_{i \in \mathcal{I}} A_i$, where $\mathcal{I} = \{1, ..., n\}$ is the index set; and $\mathbf{A}_{-i} \triangleq \prod\limits_{k \in \mathcal{I}, k \neq i} A_k$ is the Cartesian product of all sets but i. Let $\times_j f_j(.)$ be the Cartesian product of a set of functions $f_j(.)$, where $f_j : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}$, $\mathbf{z} \mapsto f_j(\mathbf{z})$, $j = 1, ..., m$. The notation $f \in \mathcal{C}^s$ is used to say that f is continuously (resp. twice continuously) differentiable when $s = 1$ (resp. $s = 2$). $\mathbf{F}$ is a m dimensional vector function when $\mathbf{F} : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$, $\mathbf{F}(\mathbf{z}) = (f_1(\mathbf{z}), ..., f_j(\mathbf{z}), ..., f_m(\mathbf{z}))$. The notation $\mathbf{z}(\mathbf{e})$, where $\mathbf{e} \in \mathbb{R}^k$, means that each z_i is a function of $\mathbf{e}$, $i = 1, ..., n$. The Jacobian matrix of $\mathbf{F}(\mathbf{z})$ with respect to $\mathbf{z}$ at $\bar{\mathbf{z}}$ is $\mathcal{J}_{\mathbf{F}_\mathbf{z}}(\bar{\mathbf{z}}) = \left[\frac{\partial(f_1, ..., f_j, ..., f_m)}{\partial(z_1, ..., z_i, ..., z_n)}(\bar{\mathbf{z}})\right]$. Let $|\mathcal{J}_{\mathbf{F}_\mathbf{z}}(\bar{\mathbf{z}})|$ be the determinant of $\mathcal{J}_\mathbf{F}$ at $\bar{\mathbf{z}}$. The Hessian matrix of $f(\mathbf{z})$ at $\bar{\mathbf{z}}$ is $\mathcal{H}_{f_\mathbf{z}}(\bar{\mathbf{z}}) = \left[\frac{\partial^2 f}{\partial z_i \partial z_j}_{|\mathbf{z}=\bar{\mathbf{z}}}\right]$, $i, j = 1, ..., n$. The bordered Hessian of $f(\mathbf{z})$ at $\bar{\mathbf{z}}$ is $\bar{\mathcal{H}}_{f_\mathbf{z}}(\bar{\mathbf{z}})$. Its determinant is $\left|\bar{\mathcal{H}}_{f_\mathbf{z}}(\bar{\mathbf{z}})\right|$. Finally, if we partition $\mathbf{z}$ such as $\mathbf{z} = (\mathbf{x}, \mathbf{y})$, then $\mathcal{J}_{\mathbf{F}_\mathbf{x}}(\bar{\mathbf{z}})$ is the Jacobian matrix of $\mathbf{F}(\mathbf{z})$ at $\bar{\mathbf{z}}$ when the differentiation is partial and made with respect to $\mathbf{x}$ only.

2. THE MODEL

Consider an exchange economy, $\mathcal{E}$, with two divisible homogeneous commodities labeled X and Y. Let p_X and p_Y be their unit prices. We assume that commodity Y is the *numéraire*, i.e. $p_Y = 1$. Traders are of two types, namely X and Y, with n_X traders of type X and n_Y traders of type Y. We assume there are m_X leaders of type X, with $m_X \geqslant 1$, and $n_X - m_X$, followers of type X, with $n_X - m_X \geqslant 1$, where $T_X := \{1, ..., m_X, m_X + 1, ..., n_X\}$. Likewise, we have $T_Y := \{1, ..., m_Y, m_Y + 1, ..., n_Y\}$, with $m_Y \geqslant 1$ and $n_Y - m_Y \geqslant 1$. Traders who belong to the set T_X (resp. T_Y) are indexed by i (resp. by j).

2.1. Assumptions on endowments and preferences

We now provide two kinds of assumptions regarding the fundamentals for $\mathcal{E}$, namely resources endowments and preferences. First, there are fixed initial endowments which satisfy the following assumption.

ASSUMPTION 1. For each $i \in T_X$, $\mathbf{w}_i = (\alpha_i, 0)$, with $\alpha_i > 0$; and, for each $j \in T_Y$, $\mathbf{w}_j = (0, \beta_j)$, with $\beta_j > 0$.

Assumption 1 is standard in the finite bilateral oligopoly game. Indeed, as emphasized by Cordella and Gabszewicz (1998), it does not require the initial endowments to be strictly in the interior of the commodity space (Amir et al. 1990), or the traders sell their entire endowments (Shubik 1973; Shapley 1976). Such distribution of endowments could echo specialization in production's technology.

Second, the preferences of each trader k are described by an utility function $u_k : \mathbb{R}^2_+ \to \mathbb{R}$, $\mathbf{z}_k \mapsto u_k(\mathbf{z}_k)$, with $\mathbf{z}_k = (x_k, y_k)$, and where x_k and y_k are the amounts of goods X and Y consumed by trader k, $k = i, j$. We make the following set of assumptions, which we designate as Assumption 2.

ASSUMPTION 2. For all $\mathbf{z}_k \in \mathbb{R}^2_+$, the utility function $u_k(\mathbf{z}_k)$ satisfies:
2a. $\forall k$, $u_k(\mathbf{z}_k) \in \mathcal{C}^2(\mathbb{R}^2_{++}, \mathbb{R})$;
2b. $\forall k$, $\frac{\partial u_k(\mathbf{z}_k)}{\partial x_k} > 0$ and $\frac{\partial u_k(\mathbf{z}_k)}{\partial y_k} > 0$;
2c. $\forall k$, $\left|\begin{bmatrix} 0 & \frac{\partial u_k}{\partial x_k} \\ \frac{\partial u_k}{\partial y_k} & \frac{\partial^2 u_k}{(\partial x_k)^2} \end{bmatrix}\right| < 0$, and $\left|\begin{bmatrix} 0 & \frac{\partial u_k}{\partial x_k} & \frac{\partial u_k}{\partial y_k} \\ \frac{\partial u_k}{\partial x_k} & \frac{\partial^2 u_k}{(\partial x_k)^2} & \frac{\partial^2 u_k}{\partial x_k \partial y_k} \\ \frac{\partial u_k}{\partial y_k} & \frac{\partial^2 u_k}{\partial y_k \partial x_k} & \frac{\partial^2 u_k}{(\partial y_k)^2} \end{bmatrix}\right| > 0$.
2d. $\lim_{x_k \to 0} \frac{\partial u_k(\mathbf{z}_k)}{\partial x_k} = \infty$ and $\lim_{y_k \to 0} \frac{\partial u_k(\mathbf{z}_k)}{\partial y_k} = \infty$ for at least one leader and one follower of each type.

Hypothesis 2a says the utility functions are twice continuously differentiable in the interior of the commodity space. And it includes the case of infinite partial derivatives along the boundary of the consumption set. 2b says the utility functions are strictly monotonic, and 2c that they are strictly quasi-concave. From 2d, the indifference curves of (at least) one trader of each type may be asymptotic to the boundary of the consumption set (for instance when $u(x, y) = \ln x + \ln y$) or may have infinite/null slope near the quantity axis (for instance when $u(x, y) = \sqrt{x} + \sqrt{y}$). These assumptions are discussed in Section 4.

2.2. The associated game

We introduce the non-cooperative strategic market game $\mathbf{\Gamma}$ associated with $\mathcal{E}$. The two-stage game $\mathbf{\Gamma}$ embodies two simultaneous move subgames, namely $\mathbf{\Gamma}_L$ and $\mathbf{\Gamma}_F$. The $m_X + m_Y$ leaders (resp. $(n_X - m_X + n_Y - m_Y)$ followers) compete in the leader-level game $\mathbf{\Gamma}_L$ (resp. follower-level game $\mathbf{\Gamma}_F$). We assume the timing of positions is given. No trader makes a choice in two subgames. In addition, traders meet once and cannot make binding agreements. By precluding such agreements, we consider each trader acts independently and without communication with any of the others. Information is assumed to be complete. Moreover, information is imperfect in each subgame, i.e. in the leader-level (resp. follower-level) game.

The traders can offer only a fraction of the commodity they initially hold. Thus, by contracting her offer, each trader manipulates the relative price. Let $\mathcal{S}_i$ be the strategy set of leader $i \in T_X$ and $\mathcal{S}_j$ be the strategy set of leader $j \in T_Y$, with:

$$\mathcal{S}_i := \{q_i \in \mathbb{R}_+ : q_i \leqslant \alpha_i\},\ i = 1, ..., m_X, \tag{1}$$

$$\mathcal{S}_j := \{b_j \in \mathbb{R}_+ : b_j \leqslant \beta_j\},\ j = 1, ..., m_Y. \tag{2}$$

The quantity q_i in (1) is the pure strategy of trader $i \in T_X$: it represents the amount of commodity X leader i offers in exchange for commodity Y. Likewise, b_j is the pure strategy of leader $j \in T_Y$. Let $\mathbf{S}^L := \prod_{i=1}^{m_X} \mathcal{S}_i \times \prod_{j=1}^{m_Y} \mathcal{S}_j$ and let $(\mathbf{q}^L; \mathbf{b}^L) \in \mathbf{S}^L$ denote the strategy profile for the leaders, that is $(\mathbf{q}^L; \mathbf{b}^L) = (q_1, ..., q_{m_X}; b_1, ..., b_{m_Y})$. Given $(\mathbf{q}^L; \mathbf{b}^L) \in \mathbf{S}^L$, the followers' strategy sets are given by:

$$\mathcal{S}_i := \{q_i(\mathbf{q}^L; \mathbf{b}^L) : \mathcal{S}^L \to [0, \alpha_i]\},\ i = m_X + 1, ..., n_X, \tag{3}$$

$$\mathcal{S}_j := \{b_j(\mathbf{q}^L; \mathbf{b}^L) : \mathcal{S}^L \to [0, \beta_j]\},\ j = m_Y + 1, ..., n_Y. \tag{4}$$

Let $\mathbf{S}^F := \prod_{i=m_X+1}^{n_X} \mathcal{S}_i \times \prod_{j=m_Y+1}^{n_Y} \mathcal{S}_j$, and let $(\mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L)) \in \mathbf{S}^F$ denote the strategy profile for the followers, that is $(\mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L)) = (q_{m_X+1}(\mathbf{q}^L; \mathbf{b}^L), ..., q_{n_X}(\mathbf{q}^L; \mathbf{b}^L); b_{m_Y+1}(\mathbf{q}^L; \mathbf{b}^L), ..., b_{n_Y}(\mathbf{q}^L; \mathbf{b}^L))$.

A strategy profile for the traders is a vector $(\mathbf{q}^L, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^L, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L)) \in \mathbf{S}$, where $\mathbf{S} := \prod_{i\in T_X} \mathcal{S}_i \times \prod_{j \in T_Y} \mathcal{S}_j$. To lighten notations, in what follows, let $(\mathbf{q}^L, \mathbf{q}^F(.); \mathbf{b}^L, \mathbf{b}^F(.))$ for $(\mathbf{q}^L, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^L, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L))$.

For each $(\mathbf{q}^L, \mathbf{q}^F(.); \mathbf{b}^L, \mathbf{b}^F(.)) \in \mathbf{S}$, the price vector $(p_X(\mathbf{q}^L, \mathbf{q}^F(.); \mathbf{b}^L, \mathbf{b}^F(.)), 1)$ is determined according to the following price mechanism which aggregates the strategic supplies of all traders:

$$p_X(\mathbf{q}^L, \mathbf{q}^F(.); \mathbf{b}^L, \mathbf{b}^F(.)) = \left\{ \begin{array}{c} \frac{B}{Q}, \text{ if } B > 0 \text{ and } Q > 0, \\ 0, \text{ otherwise,} \end{array} \right. \tag{5}$$

where $Q \equiv \sum_{i=1}^{m_X} q_i + \sum_{i=m_X+1}^{n_X} q_i(\mathbf{q}^L; \mathbf{b}^L)$ and $B \equiv \sum_{j=1}^{m_Y} b_j + \sum_{j=m_Y+1}^{n_Y} b_j(\mathbf{q}^L; \mathbf{b}^L)$. With a slight abuse of notations, in what follows, let $p_X \triangleq p_X(\mathbf{q}^L, \mathbf{q}^F(.); \mathbf{b}^L, \mathbf{b}^F(.))$.

The final allocation received by each trader for the good for which s/he has an allocation is the amount s/he has kept after trade, and the final allocation received of the other good is proportional to the quantity s/he sells. Leader $i \in T_X$ obtains in exchange for q_i a share $\frac{q_i}{Q}$ of the aggregate supply B, i.e. a quantity of commodity Y

equal to $p_X q_i$, and ends up with the bundle of commodities $(x_i(q_i, p_X), y_i(q_i, p_X)) = (1 - q_i \ \ , p_X q_i)$. Her corresponding utility level is $u_i(1 - q_i, p_X q_i)$. Likewise, follower $i \in T_X$, by supplying $q_i(\mathbf{q}^L; \mathbf{b}^L)$, ends up with the bundle of commodities $(x_i(q_i(\mathbf{q}^L; \mathbf{b}^L), p_X), y_i(q_i(\mathbf{q}^L; \mathbf{b}^L), p_X)) = (1 - q_i(\mathbf{q}^L; \mathbf{b}^L) \ \ , p_X q_i(\mathbf{q}^L; \mathbf{b}^L))$, and reaches the utility level $u_i(1 - q_i(\mathbf{q}^L; \mathbf{b}^L), p_X q_i(\mathbf{q}^L; \mathbf{b}^L))$. Leader $j \in T_Y$ obtains in exchange for b_j a share $\frac{b_j}{B}$ of the aggregate supply Q, i.e. a quantity of commodity X equal to $\frac{1}{p_X} b_j$, and ends up with the bundle $(x_j(b_j, p_X), y_j(b_j, p_X)) = (\frac{1}{p_X} b_j, 1 - b_j)$, with utility level $u_j(\frac{1}{p_X} b_j, 1 - b_j)$. Likewise, follower $j \in T_Y$, by supplying $b_j(\mathbf{q}^L; \mathbf{b}^L)$, ends up with the bundle $(x_j(b_j(\mathbf{q}^L; \mathbf{b}^L), p_X), y_j(b_j(\mathbf{q}^L; \mathbf{b}^L), p_X)) = (\frac{1}{p_X} b_j(\mathbf{q}^L; \mathbf{b}^L), 1 - b_j(\mathbf{q}^L; \mathbf{b}^L))$, and reaches the utility level $u_j(\frac{1}{p_X} b_j(\mathbf{q}^L; \mathbf{b}^L), 1 - b_j(\mathbf{q}^L; \mathbf{b}^L))$. Therefore, the final allocations assign the following bundles for leaders:

$$\mathbf{z}_i(q_i, p_X) = \{ \begin{array}{c} (\alpha_i - q_i, p_X q_i), \text{ if } p_X > 0 \\ (\alpha_i, 0), \text{ if } p_X = 0 \end{array}, \ i = 1, ..., m_X, \tag{6}$$

$$\mathbf{z}_j(b_j, p_X) = \{ \begin{array}{c} (\frac{1}{p_X} b_j, \beta_j - b_j), \text{ if } p_X > 0 \\ (0, \beta_j), \text{ if } p_X = 0. \end{array}, \ j = 1, ..., m_Y, \tag{7}$$

and the following bundles for followers $i \in \{m_X+1, ..., n_X\}$ and $j \in \{m_Y+1, ..., n_Y\}$:

$$\mathbf{z}_i(q_i(\mathbf{q}^L; \mathbf{b}^L), p_X) = \{ \begin{array}{c} (\alpha_i - q_i(\mathbf{q}^L; \mathbf{b}^L), p_X q_i(\mathbf{q}^L; \mathbf{b}^L)), \text{ if } p_X > 0 \\ (\alpha_i, 0), \text{ if } p_X = 0 \end{array}, \tag{8}$$

$$\mathbf{z}_j(b_j(\mathbf{q}^L; \mathbf{b}^L), p_X) = \{ \begin{array}{c} (\frac{1}{p_X} b_j(\mathbf{q}^L; \mathbf{b}^L), \beta_j - b_j(\mathbf{q}^L; \mathbf{b}^L)), \text{ if } p_X > 0 \\ (0, \beta_j), \text{ if } p_X = 0. \end{array}. \tag{9}$$

Finally, let us define the payoffs of traders. Define the function $\pi_i : \mathbf{S} \to \mathbb{R}$, $(q_i, \mathbf{q}^L_{-i}, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^L, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L)) \mapsto \pi_i(q_i, \mathbf{q}^L_{-i}, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^L, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L))$, for each $i = 1, ..., m_X$. Likewise, let $\pi_j : \mathbf{S} \to \mathbb{R}$, $(\mathbf{q}^L, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); b_j, \mathbf{b}^L_{-j}, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L)) \mapsto \pi_j(\mathbf{q}^L, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); b_j, \mathbf{b}^L_{-j}, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L))$, $j = 1, ..., m_Y$. To lighten notations, let $\pi_i(q_i, .)$, $i = 1, ..., m_X$, and $\pi_j(b_j, .)$, $j = 1, ..., m_Y$. Then, the utility levels of leaders may be written as payoffs:

$$\pi_i(q_i, .) = u_i(\alpha_i - q_i, \frac{\sum_{j=1}^{m_Y} b_j + \sum_{j=m_Y+1}^{n_Y} b_j(\mathbf{q}^L; \mathbf{b}^L)}{q_i + \sum_{k=1, k \neq i}^{m_X} q_k + \sum_{k=m_X+1}^{n_X} q_k(\mathbf{q}^L; \mathbf{b}^L)} q_i), \ i = 1, ..., m_X, \tag{10}$$

$$\pi_j(b_j, .) = u_j(\frac{\sum_{i=1}^{m_X} q_i + \sum_{i=m_X+1}^{n_X} q_i(\mathbf{q}^L; \mathbf{b}^L)}{b_j + \sum_{l=1, l \neq j}^{m_Y} b_l + \sum_{l=m_Y+1}^{n_Y} b_l(\mathbf{q}^L; \mathbf{b}^L)} b_j, \beta_j - b_j), \ j = 1, ..., m_Y. \tag{11}$$

And, define the function $\pi_i : \mathbf{S} \to \mathbb{R}$, $(q_i(\mathbf{q}^L; \mathbf{b}^L), \mathbf{q}^F_{-i}(\mathbf{q}^L; \mathbf{b}^L), \mathbf{q}^L; \mathbf{b}^L, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L)) \mapsto \pi_i(q_i(\mathbf{q}^L; \mathbf{b}^L), \mathbf{q}^F_{-i}(\mathbf{q}^L; \mathbf{b}^L), \mathbf{q}^L; \mathbf{b}^L, \mathbf{b}^F(\mathbf{q}^L; \mathbf{b}^L))$, for each $i = m_X + 1, ..., n_X$. Likewise, define the function $\pi_j : \mathbf{S} \to \mathbb{R}$, $(\mathbf{q}^L, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^L, b_j(\mathbf{q}^L; \mathbf{b}^L), \mathbf{b}^F_{-j}(\mathbf{q}^L; \mathbf{b}^L)) \mapsto \pi_j(\mathbf{q}^L, \mathbf{q}^F(\mathbf{q}^L; \mathbf{b}^L); \mathbf{b}^L, b_j(\mathbf{q}^L; \mathbf{b}^L), \mathbf{b}^F_{-j}(\mathbf{q}^L; \mathbf{b}^L))$, for each $j = m_Y + 1, ..., n_Y$. To lighten notations, let $\pi_i(q_i(.),.)$, where $q_i(.) = q_i(\mathbf{q}^L;\mathbf{b}^L)$, for each $i = m_X + 1,...,n_X$, and $\pi_j(b_j(.),.)$, where $b_j(.) = b_j(\mathbf{q}^L;\mathbf{b}^L)$, for each $j = m_Y + 1,...,n_Y$. Then, the utility levels of followers $i \in \{m_X + 1,...,n_X\}$ and $j \in \{m_Y + 1,...,n_Y\}$ may be written as payoffs:

$$\pi_i(q_i(.),.) = u_i(\alpha_i - q_i(.), \frac{\sum\limits_{j=1}^{m_Y} b_j + \sum\limits_{j=m_Y+1}^{n_Y} b_j(.)}{q_i(.) + \sum\limits_{k=1}^{m_X} q_k + \sum\limits_{k=m_X+1,k\neq i}^{n_X} q_k(.)} q_i(.)), \tag{12}$$

$$\pi_j(b_j(.),.) = u_j(\frac{\sum\limits_{i=1}^{m_X} q_i + \sum\limits_{i=m_X+1}^{n_X} q_i(.)}{b_j(.) + \sum\limits_{l=1,l\neq j}^{m_Y} b_l + \sum\limits_{l=m_Y+1}^{n_Y} b_l(.)} b_j(.), \beta_j - b_j(.)). \tag{13}$$

2.3. Stackelberg-Nash equilibrium: definition

We now turn to the definition of a Stackelberg-Nash equilibrium. To this end, we define some concepts that are related to the behavior of traders in each subgame.

Consider the subgame $\mathbf{\Gamma}_F$. For each $(\mathbf{q}^L;\mathbf{b}^L) \in \mathbf{S}^L$, the best reply correspondences of followers are defined as follows.

DEFINITION 1. Let $\phi_i : \mathbf{S}_{-i} \to \mathcal{S}_i$, with $\phi_i(\mathbf{q}^L, \mathbf{q}^F_{-i}; \mathbf{b}^L, \mathbf{b}^F) : = \{q_i \in \mathcal{S}_i : q_i \in \arg\max \pi_i(\mathbf{q}^L, q_i, \mathbf{q}^F_{-i}; \mathbf{b}^L, \mathbf{b}^F)\}$, be follower i's best reply correspondence, $i = m_X + 1,...,n_X$. Likewise, let $\psi_j : \mathbf{S}_{-j} \to \mathcal{S}_j$. with $\psi_j(\mathbf{q}^L, \mathbf{q}^F; \mathbf{b}^L, \mathbf{b}^F_{-j}) : = \{b_j \in \mathcal{S}_j : b_j \in \arg\max \pi_j(\mathbf{q}^L, \mathbf{q}^F; \mathbf{b}^L, b_j, \mathbf{b}^F_{-j})\}$, be follower j's best reply correspondence, $j = m_Y + 1,...,n_Y$.

It is worth noting that this game displays a rich set of strategic interactions. Therefore, with several followers, by contrast with the duopoly game in which the best reply of the follower always coincides with her strategy, the followers' best responses differ from their strategies. With several followers the best responses could be inconsistent, and thereby, the followers' strategies could not be well defined (see Example 3 in Section 4). The existence, for each follower, of a unique smooth strategy is studied in Section 3. The followers' strategies are defined as follows.

DEFINITION 2. Let $\sigma_i : \mathbf{S}^L \to \mathcal{S}_i$, with $(\mathbf{q}^L;\mathbf{b}^L) \mapsto \sigma_i(\mathbf{q}^L;\mathbf{b}^L)$, be the strategy of follower i, $i = i = m_X + 1,...,n_X$. Likewise, let $\varphi_j : \mathbf{S}^L \to \mathcal{S}_j$, with $(\mathbf{q}^L;\mathbf{b}^L) \mapsto \varphi_j(\mathbf{q}^L;\mathbf{b}^L)$., be the strategy of follower $j = m_Y + 1,...,n_Y$.

Define the correspondence $\boldsymbol{\sigma} : \mathbf{S}^L \to \prod\limits_{i=m_X+1}^{n_X} \mathcal{S}_i$, with $\mathbf{q}^F \in \boldsymbol{\sigma}(\mathbf{q}^L;\mathbf{b}^L)$, and the correspondence $\boldsymbol{\varphi} : \mathbf{S}^L \to \prod\limits_{j=m_Y+1}^{n_Y} \mathcal{S}_j$, with $\mathbf{b}^F \in \boldsymbol{\varphi}(\mathbf{q}^L;\mathbf{b}^L)$. Then, the price p_X may be written as $p_X(\mathbf{q}^L, \boldsymbol{\sigma}(\mathbf{q}^L;\mathbf{b}^L); \mathbf{b}^L, \boldsymbol{\varphi}(\mathbf{q}^L;\mathbf{b}^L))$.

Consider now the subgame $\mathbf{\Gamma}_L$, and the best reply correspondences of leaders.

DEFINITION 3. Let $\phi_i : \mathbf{S}^L_{-i} \to \mathcal{S}_i$, with $\phi_i(\mathbf{q}^L_{-i};\mathbf{b}^L) : = \{q_i \in \mathcal{S}_i : q_i \in \arg\max \pi_i(q_i, \mathbf{q}^L_{-i}, \boldsymbol{\sigma}(q_i, \mathbf{q}^L_{-i}; \mathbf{b}^L); \mathbf{b}^L, \boldsymbol{\varphi}(q_i, \mathbf{q}^L_{-i}; \mathbf{b}^L)\}$, be leader i's best reply correspondence, $i = 1,...,m_X$. Likewise, let $\psi_j : \mathbf{S}^L_{-j} \to \mathcal{S}_j$, with $\psi_j(\mathbf{q}^L;\mathbf{b}^L_{-j}) : = \{b_j \in \mathcal{S}_j : b_j \in \arg\max \pi_j(\mathbf{q}^L, \boldsymbol{\sigma}(\mathbf{q}^L; b_j, \mathbf{b}^L_{-j}); b_j, \mathbf{b}^L_{-j}, \boldsymbol{\varphi}(\mathbf{q}^L; b_j, \mathbf{b}^L_{-j})\}$, be leader j's best reply correspondence, $j = 1,...,m_Y$.

Let us now consider the consistency of optimal behaviors. The equilibrium of the two-stage game $\mathbf{\Gamma}$ is a pure strategy SPNE, while the equilibria in both subgames $\mathbf{\Gamma}_L$ and $\mathbf{\Gamma}_F$ are Nash equilibria. But such a SPNE is a Nash equilibrium (NE thereafter) of each subgame of $\mathbf{\Gamma}$ (Selten 1975).[6]

Therefore, consider the subgame $\mathbf{\Gamma}_L$, and let $(\mathbf{q}^L;\mathbf{b}^L) \in \mathbf{S}^L$. Define the family of functions $\mathbf{\Lambda}_L : \mathbf{S}^L \to \mathbf{S}^L$, with $\mathbf{\Lambda}_L(\mathbf{q}^L, \boldsymbol{\sigma}(\mathbf{q}^L;\mathbf{b}^L);\mathbf{b}^L, \boldsymbol{\varphi}(\mathbf{q}^L;\mathbf{b}^L)) = \times_{i=1}^{m_X}\phi_i \times_{j=1}^{m_Y}\psi_j$. A pure strategy NE of the subgame $\mathbf{\Gamma}_L$ is a fixed point $(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)$ of $\mathbf{\Lambda}_L(.)$ such that no leader has an interest to deviate unilaterally from her decision. Consider now the subgame $\mathbf{\Gamma}_F$, and define $\mathbf{\Lambda}_F : \mathbf{S} \to \mathbf{S}^F$, with $\mathbf{\Lambda}_F(\tilde{\mathbf{q}}^L, \mathbf{q}^F;\tilde{\mathbf{b}}^L, \mathbf{b}^F) = \times_{i=m_X+1}^{n_X}\phi_i \times_{j=m_Y+1}^{n_Y}\psi_j$, with $\mathbf{q}^F \in \boldsymbol{\sigma}(\mathbf{q}^L;\mathbf{b}^L)$ and $\mathbf{b}^F \in \boldsymbol{\varphi}(\mathbf{q}^L;\mathbf{b}^L)$. A pure strategy NE of the subgame $\mathbf{\Gamma}_F$ is a fixed point $(\tilde{\mathbf{q}}^F;\tilde{\mathbf{b}}^F)$ of $\mathbf{\Lambda}_F(.)$ such that no follower has an interest to deviate unilaterally from his decision.

Finally, consider the entire game $\mathbf{\Gamma}$. A pure strategy SPNE of $\mathbf{\Gamma}$ is a fixed point $(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)) \in \mathbf{S}$, with $(\tilde{\mathbf{q}}^F(.);\tilde{\mathbf{b}}^F(.)) \in (\boldsymbol{\sigma}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\boldsymbol{\varphi}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L))$. Then, (5) is $\tilde{p}_X = p_X(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L))$. The allocations to leaders (6)-(7) are $\tilde{\mathbf{z}}_i = \mathbf{z}_i(\tilde{q}_i, \tilde{p}_X)$, $i = 1,...,m_X$, and $\tilde{\mathbf{z}}_j = \mathbf{z}_j(\tilde{b}_j, \tilde{p}_X)$, $j = 1,...,m_Y$, and the allocations to followers (8)-(9) are $\tilde{\mathbf{z}}_i = \mathbf{z}_i(\tilde{q}_i(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{p}_X)$, $i = m_X + 1,...,n_X$, and $\tilde{\mathbf{z}}_j = \mathbf{z}_j(\tilde{b}_j(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{p}_X)$, $j = m_Y + 1,...,n_Y$. Then, the leaders' payoffs (10)-(11) are given by $\pi_i(\tilde{q}_i, \tilde{\mathbf{q}}^L_{-i}, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)) = u_i(\mathbf{z}_i(\tilde{q}_i, \tilde{p}_X))$, for each $i = 1,...,m_X$, and $\pi_j(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{b}_j, \tilde{\mathbf{b}}^L_{-j}, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)) = u_j(\mathbf{z}_j(\tilde{b}_j, \tilde{p}_X))$, for each $j = 1,...,m_Y$, and the followers' payoffs corresponding to (12)-(13) are given by $\pi_i(\tilde{\mathbf{q}}^L, \tilde{q}_i(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{\mathbf{q}}^F_{-i}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)) = u_i(\mathbf{z}_i(\tilde{q}_i(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{p}_X))$, for each $i = m_X + 1,...,n_X$, and $\pi_j(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{b}_j(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{\mathbf{b}}^F_{-j}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)) = u_j(\mathbf{z}_j(\tilde{b}_j(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{p}_X))$, for each $j = m_Y + 1,...,n_Y$.

We are now able to define formally a SNE for the game $\mathbf{\Gamma}$.

DEFINITION 4. (SNE). A $(n_X + n_Y)$-tuple $(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L))$ is a Stackelberg-Nash equilibrium of $\mathbf{\Gamma}$ if:

a. $\forall i \in \{1,...,m_X\}$ $\pi_i(\tilde{q}_i, \tilde{\mathbf{q}}^L_{-i}, \tilde{\mathbf{q}}^F(\tilde{q}_i, \tilde{\mathbf{q}}^L_{-i};\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{q}_i, \tilde{\mathbf{q}}^L_{-i};\tilde{\mathbf{b}}^L)) \geqslant$
$$\pi_i(q_i, \tilde{\mathbf{q}}^L_{-i}, \mathbf{q}^F(q_i, \tilde{\mathbf{q}}^L_{-i};\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \mathbf{b}^F(q_i, \tilde{\mathbf{q}}^L_{-i};\tilde{\mathbf{b}}^L)),$$
for all $\mathbf{q}^F(\mathbf{q}^L;\mathbf{b}^L) \in \prod_{i=m_X+1}^{n_X} \mathcal{S}_i$ and all $\mathbf{b}^F(\mathbf{q}^L;\mathbf{b}^L) \in \prod_{j=m_Y+1}^{n_Y} \mathcal{S}_j$, for all $q_i \in \mathcal{S}_i$;

b. $\forall j \in \{1,...,m_Y\}$ $\pi_j(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{b}_j, \tilde{\mathbf{b}}^L_{-j});\tilde{b}_j, \tilde{\mathbf{b}}^L_{-j}, \tilde{\mathbf{b}}^L(\tilde{\mathbf{q}}^L;\tilde{b}_j, \tilde{\mathbf{b}}^L_{-j})) \geqslant$
$$\pi_j(\tilde{\mathbf{q}}^L, \mathbf{q}^F(\tilde{\mathbf{q}}^L;b_j, \tilde{\mathbf{b}}^L_{-j});b_j, \tilde{\mathbf{b}}^L_{-j}, \mathbf{b}^F(\tilde{\mathbf{q}}^L;b_j, \tilde{\mathbf{b}}^L_{-j})),$$
for all $\mathbf{q}^F(\mathbf{q}^L;\mathbf{b}^L) \in \prod_{i=m_X+1}^{n_X} \mathcal{S}_i$ and all $\mathbf{b}^F(\mathbf{q}^L;\mathbf{b}^L) \in \prod_{j=m_Y+1}^{n_Y} \mathcal{S}_j$, for all $b_j \in \mathcal{S}_j$;

c. $\forall i \in \{m_X + 1,...,n_X\}$ $\pi_i(\tilde{\mathbf{q}}^L, \tilde{q}_i(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{\mathbf{q}}^F_{-i}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)) \geqslant \pi_i(\tilde{\mathbf{q}}^L, q_i(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{\mathbf{q}}^F_{-i}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L))$, for all $q_i \in \mathcal{S}_i$;

d. $\forall j \in \{m_Y + 1,...,n_Y\}$ $\pi_j(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, \tilde{b}_j(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{\mathbf{b}}^F_{-j}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)) \geqslant \pi_j(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L, b_j(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L), \tilde{\mathbf{b}}^F_{-j}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L))$, for all $b_j \in \mathcal{S}_j$.

Therefore, a SNE is a noncooperative oligopoly equilibrium of $\mathbf{\Gamma}$ such that, in each stage of the game, no trader has interest to deviate unilaterally from her choice.

[6] It requires the strategies of the leaders and the followers to constitute a NE of any subgame. In addition, it is a SPNE without empty threats: it rules out incredible threats by the followers. The reason is the strategy of any follower is optimal for any supply set by the leaders. The followers can set their own supplies according to any possible function of the quantities set by the leaders, with the belief that the leaders will not counter-react. Similarly, the leaders expect the followers to conform to the decisions given by their best responses.

3. EXISTENCE OF A SNE WITH TRADE

Let us now turn to the existence of a SNE with trade. It is well known that the autarkic equilibrium is always a Nash equilibrium (NE) in simultaneous move strategic market games (Cordella and Gabszewicz 1998; Giraud 2003; Busetto and Codognato 2006). The following example, which is borrowed from Cordella and Gabszewicz (1998), and adapted to our setting, illustrates this feature in the sequential game.

EXAMPLE 1. (Autarkic SNE). Let $\#T_1 = \#T_2 = 2$. Assumption 1 is $\alpha_i = 1$, for each $i \in T_X$, and $\beta_j = 1$, for each $j \in T_Y$. Assumption 2 is $u_i(x_i, y_i) = \gamma x_i + y_i$, $i \in T_X$, and $u_i(x_i, y_i) = x_i + \gamma y_i$, $j \in T_Y$, with $\gamma \in (0, 1)$, so (2d) does not hold. The unique competitive equilibrium is given by $p^*_X = 1$ and $z^*_i = (0, 1)$, $i \in T_X$, and $z^*_j = (1, 0)$, $j \in T_Y$. In addition, the Cournot-Nash equilibrium strategies are given by $(\hat{q}_1, \hat{q}_2; \hat{b}_1, \hat{b}_2) = (0, 0; 0, 0)$. Consider now the SNE. The followers' best responses are $\phi_2(q_1, q_2; b_1) = -b_1 + \sqrt{\frac{1}{\gamma} b_1(q_1 + q_2)}$ and $\psi_2(q_1; b_1, b_2) = -q_1 + \sqrt{\frac{1}{\gamma}(b_1 + b_2)q_1}$. The strategies are given by $\sigma(q_1; b_1) = \frac{(1-2\gamma)b_1 + \sqrt{(1-4\gamma)(b_1)^2 + 4\gamma b_1 q_1}}{2\gamma}$ and $\varphi(q_1; b_1) = \frac{(1-2\gamma)q_1 + \sqrt{(1-4\gamma)(q_1)^2 + 4\gamma b_1 q_1}}{2\gamma}$. Then, the leaders' SNE strategies are $\tilde{q}_1 = 0$ and $\tilde{b}_1 = 0$. Accordingly, the strategies of followers are $\sigma(0; 0) = 0$ and $\varphi(0; 0) = 0$. Then, the only SNE is the trivial equilibrium $(\tilde{q}_1, \tilde{q}_2; \tilde{b}_1, \tilde{b}_2) = (0, 0; 0, 0)$.

Therefore, if all traders but one are making a null supply, any other trader, whichever her type is, will not deviate by making a positive supply. Indeed, no leader/follower finds it profitable to participate in exchange as long as no other leader/follower does. For any trader, and whichever is the stage of the game, the strategic advantage from trading is offset by the strategic advantage of reducing her supply. Does this imply that there is never a non-trivial sequential equilibrium? The following theorem provides a negative answer to this question.

THEOREM (Existence of a SNE with trade). *Consider the finite game* $\mathbf{\Gamma}$, *and let Assumptions 1 and 2 be satisfied. Then, there exists a Stackelberg-Nash equilibrium with trade.*

The remaining part of this section is devoted to the proof of the theorem. To prove the theorem, we consider a slight perturbation of the strategic market game as in Dubey and Shubik (1978) when they show existence of non-autarkic Cournot-Nash equilibria. Our proof requires five steps. First, we study the optimal behaviors in each perturbed subgame. This leads to study the followers's best responses, and then to show the existence of unique smooth followers' strategies in the perturbed game. Then, by considering the subgame between leaders, we study the leaders's best responses. Second, we prove the existence of a SNE of the perturbed game, i.e., we determine the conditions under which the best responses of traders are mutually consistent in each perturbed subgame as well as in the entire perturbed game. Third, we show that there exist some uniform bounds on the market price in each perturbed subgame. Fourth, we prove the SNE of the perturbed game is non-autarkic. Fifth, we show that the SNE with trade is an equilibrium point of the game, i.e., a non trivial subgame perfect Nash equilibrium which is robust to slight perturbation of the game.

Therefore, consider a slight perturbation of the game $\mathbf{\Gamma}$. Let $\mathbf{\Gamma}^\epsilon$ be the perturbed game in which some outside agency puts a fixed quantity $\epsilon > 0$ of the two commodities on each side of the market. This does not change the traders' strategy sets, but it changes outcomes and payoffs. Given $\epsilon > 0$, the price (5) of $\mathbf{\Gamma}^\epsilon$ is now given by:

$$p_X^\epsilon = \frac{B+\epsilon}{Q+\epsilon}. \tag{14}$$

To lighten notations, let $\mathbf{q}_\epsilon^F(.)$ for $\mathbf{q}_\epsilon^F(\mathbf{q}_\epsilon^L;\mathbf{b}_\epsilon^L)$ and $\mathbf{b}_\epsilon^L(.)$ for $\mathbf{b}_\epsilon^F(\mathbf{q}_\epsilon^L;\mathbf{b}_\epsilon^L)$. Then, let $p_X^\epsilon \triangleq p_X(\mathbf{q}_\epsilon^L,\mathbf{q}_\epsilon^F(.);\mathbf{b}_\epsilon^L,\mathbf{b}_\epsilon^F(.))$. Therefore, the allocations in $\mathbf{\Gamma}^\epsilon$ are given by $\mathbf{z}_{i,\epsilon} = \mathbf{z}_{i,\epsilon}(q_{i,\epsilon},p_X^\epsilon)$, for each $i = 1,...,m_X$, and $\mathbf{z}_{j,\epsilon} = \mathbf{z}_{j,\epsilon}(b_{j,\epsilon},p_X^\epsilon)$, for each $j = 1,...,m_Y$, and by $\mathbf{z}_{i,\epsilon} = \mathbf{z}_{i,\epsilon}(q_{i,\epsilon}(.),p_X^\epsilon)$, for each $i = m_X+1,...,n_X$, and $\mathbf{z}_{j,\epsilon} = \mathbf{z}_{j,\epsilon}(b_{j,\epsilon}(.),p_X^\epsilon)$, for each $j = m_Y+1,...,n_Y$. The payoffs in $\mathbf{\Gamma}^\epsilon$ are given by $\pi_{i,\epsilon}^\epsilon(q_{i,\epsilon},\mathbf{q}_{-i,\epsilon}^L,\mathbf{q}_\epsilon^F(.);\mathbf{b}_\epsilon^L,\mathbf{b}_\epsilon^F(.)) = u_i(\mathbf{z}_{i,\epsilon}(q_{i,\epsilon},p_X^\epsilon))$, for each $i = 1,...,m_X$, and $\pi_j^\epsilon(\mathbf{q}_\epsilon^L,\mathbf{q}_\epsilon^F(.);b_{j,\epsilon},\mathbf{b}_{-j,\epsilon}^L,\mathbf{b}_\epsilon^F(.)) = u_j(\mathbf{z}_{j,\epsilon}(b_{j,\epsilon},p_X^\epsilon))$, for each $j = 1,...,m_Y$, and by $\pi_i^\epsilon(\mathbf{q}_\epsilon^L,q_{i,\epsilon}(.),\mathbf{q}_{-i,\epsilon}^F(.);\mathbf{b}_\epsilon^L,\mathbf{b}_\epsilon^F(.)) = u_i(\mathbf{z}_{i,\epsilon}(q_{i,\epsilon}(.),p_X^\epsilon))$, for each $i = m_X+1,...,n_X$, and $\pi_j^\epsilon(\mathbf{q}_\epsilon^L,\mathbf{q}_\epsilon^F(.);\mathbf{b}_\epsilon^L,b_{j,\epsilon}(.),\mathbf{b}_{-j}^F(.)) = u_j(\mathbf{z}_{j,\epsilon}(b_{j,\epsilon}(.),p_X^\epsilon))$, for each $j = m_Y+1,...,n_Y$. We now define formally the concept of ϵ-SNE.

DEFINITION 5 (ϵ-SNE). For all $\epsilon > 0$, a (n_X+n_Y)-tuple $(\tilde{\mathbf{q}}_\epsilon^L,\tilde{\mathbf{q}}_\epsilon^F(.);\tilde{\mathbf{b}}_\epsilon^L,\tilde{\mathbf{b}}_\epsilon^F(.))$ is a Stackelberg-Nash equilibrium of $\mathbf{\Gamma}^\epsilon$ if conditions a., b., c. and d. in Definition 4 hold, but where π_i is replaced by π_i^ϵ for each $i \in \{1,...,n_X\}$, and π_j is replaced by π_j^ϵ for each $j \in \{1,...,n_Y\}$ respectively.

To show the existence of an ϵ-SNE (with trade) we need some intermediate results. First, we consider the behavior of traders in the perturbed game $\mathbf{\Gamma}^\epsilon$.

Consider the perturbed subgame $\mathbf{\Gamma}_F^\epsilon$. The problem of follower i (resp. j) consists of maximizing his payoff $\pi_i^\epsilon(q_{i,\epsilon},\mathbf{q}_\epsilon^L,\mathbf{q}_{-i,\epsilon}^F;\mathbf{b}_\epsilon^L,\mathbf{b}_\epsilon^F)$ (resp. $\pi_j^\epsilon(\mathbf{q}_\epsilon^L,\mathbf{q}_\epsilon^F;b_{j,\epsilon},\mathbf{b}_\epsilon^L,\mathbf{b}_{-j,\epsilon}^F)$). The next proposition echoes Definition 1.

PROPOSITION 1. *Let Assumption 2 be satisfied. Then, for all* $\epsilon > 0$, *the best responses* $\phi_i^\epsilon : \mathbf{S}_{-i}\times\mathbb{R}_{++} \to \mathcal{S}_i$, *with* $(\mathbf{q}_\epsilon^L,\mathbf{q}_{-i,\epsilon}^F;\mathbf{b}_\epsilon^L,\mathbf{b}_\epsilon^F;\epsilon) \mapsto \phi_i^\epsilon(\mathbf{q}_\epsilon^L,\mathbf{q}_{-i,\epsilon}^F;\mathbf{b}_\epsilon^L,\mathbf{b}_\epsilon^F;\epsilon)$, $i = m_X+1,...,n_X$, and $\psi_j : \mathbf{S}_{-j}\times\mathbb{R}_{++} \to \mathcal{S}_j$, *with* $(\mathbf{q}_\epsilon^L,\mathbf{q}_\epsilon^F;\mathbf{b}_\epsilon^L,\mathbf{b}_{-j,\epsilon}^F;\epsilon) \mapsto \psi_j^\epsilon(\mathbf{q}_\epsilon^L,\mathbf{q}_\epsilon^F;\mathbf{b}_\epsilon^L,\mathbf{b}_{-j,\epsilon}^F;\epsilon)$, $j = m_Y+1,...,n_Y$ *exist and are twice-continuously differentiable functions.*

PROOF. See Appendix A.

The next proposition says that the followers' best responses are bounded.

PROPOSITION 2. *Let* $\boldsymbol{\phi}^\epsilon = (\phi_{m_X+1}^\epsilon,...,\phi_{n_X}^\epsilon)$ *and* $\boldsymbol{\psi}^\epsilon = (\psi_{m_Y+1}^\epsilon,...,\psi_{n_Y}^\epsilon)$ *be respectively* (n_X-m_X) *and* (n_Y-m_Y) *dimensional vector functions. Let* $\bar{\mathbf{a}} = (\bar{\mathbf{q}}_\epsilon^L,\bar{\mathbf{q}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L;\bar{\mathbf{b}}_\epsilon^L);\bar{\mathbf{b}}_\epsilon^L,\bar{\mathbf{b}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L;\bar{\mathbf{b}}_\epsilon^L)) \in \mathbf{S}$. *Consider the Jacobian matrices* $\mathcal{J}_{\boldsymbol{\phi}^\epsilon_{\mathbf{q}_\epsilon^F}}(\bar{\mathbf{a}}) = \left[\frac{\partial\boldsymbol{\phi}^\epsilon(.)}{\partial\mathbf{q}_\epsilon^F}\right]$ *and* $\mathcal{J}_{\boldsymbol{\psi}^\epsilon_{\mathbf{b}_\epsilon^F}}(\bar{\mathbf{a}}) = \left[\frac{\partial\boldsymbol{\psi}^\epsilon(.)}{\partial\mathbf{b}_\epsilon^F}\right]$. *Then,* $-\mathbf{I} << \mathcal{J}_{\boldsymbol{\phi}^\epsilon_{\mathbf{q}_\epsilon^F}}(\bar{\mathbf{a}}) << \mathbf{I}$, *where* $\mathbf{I}$ *is the* (n_X-m_X,n_X-n_X) *unit matrix, and* $-\mathbf{I} << \mathcal{J}_{\boldsymbol{\psi}^\epsilon_{\mathbf{b}_\epsilon^F}}(\bar{\mathbf{a}}) << \mathbf{I}$, *where* $\mathbf{I}$ *is the* (n_Y-m_Y,n_Y-m_Y) *unit matrix. In addition,* $\mathcal{J}_{\boldsymbol{\phi}^\epsilon_{\mathbf{b}_\epsilon^F}}(\bar{\mathbf{a}}) = \left[\frac{\partial\boldsymbol{\phi}^\epsilon(.)}{\partial\mathbf{b}_\epsilon^F}\right] \in (-\mathbf{I},\mathbf{I})$, *and* $\mathcal{J}_{\boldsymbol{\psi}^\epsilon_{\mathbf{q}_\epsilon^F}}(\bar{\mathbf{a}}) = \left[\frac{\partial\boldsymbol{\psi}^\epsilon(.)}{\partial\mathbf{q}_\epsilon^F}\right] \in (-\mathbf{I},\mathbf{I})$, *where the* $\mathbf{I}$*s are* (n_X-m_X,n_Y-m_Y) *and* (n_Y-m_Y,n_X-m_X) *unit matrices.*

PROOF. See Appendix B.

To define the strategies of followers (see Definition 2), the followers' best responses must be consistent. By "consistent" we mean that the followers' strategies are deduced from the collection of best responses. It is worth noting that the strategies might not exist even if the best responses exist (see Example 3). To this end, we give a sufficient condition which guarantees the existence of followers' strategies.

To introduce this condition, define the function $\Phi_i^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathcal{S}_i$, with $\Phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) := q_{i,\epsilon}(.) - \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon)$, for $i = m_X + 1, ..., n_X$, and the function $\Psi_j^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathcal{S}_j$, with $\Psi_j^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) := b_{j,\epsilon}(.) - \psi_j^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_{-j,\epsilon}^F(.); \epsilon)$, for $j = m_Y + 1, ..., n_Y$, $\epsilon > 0$ (remind (.) means $(\mathbf{q}^L; \mathbf{b}^L)$). This set of functions will be useful to build the system of equations that will implicitly define the strategies.

Let $\boldsymbol{\Upsilon}^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathcal{S}^F$ the $(n_X - m_X + n_Y - m_Y)$-dimensional vector function given by $\boldsymbol{\Upsilon}^\epsilon(.) = (\Phi_{m_X+1}^\epsilon, ..., \Phi_{n_X}^\epsilon; \Psi_{m_Y+1}^\epsilon, ..., \Psi_{n_Y}^\epsilon)$. Consider the vector equation $\boldsymbol{\Upsilon}^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) = \mathbf{0}$ with $(n_X - m_X + n_Y - m_Y)$ unknowns $(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)$ and $m_X + m_Y$ parameters $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)$. This system defines implicitly (at least locally) the followers' strategies. Let $\bar{\mathbf{a}} = (\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L); \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L))$ be an interior point of $\mathbf{S}$, so the identity $\boldsymbol{\Upsilon}^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)) \equiv \mathbf{0}$ holds in an open neighborhood of $(\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L); \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L))$. Implicit partial differentiation with respect to $(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L)$ of this identity leads to:

$$\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{a}}).\mathcal{A}^\epsilon = -\mathcal{B}^\epsilon, \text{ for each } \varepsilon > 0, \tag{15}$$

where $\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{a}})$ is the $(n_X - m_X + n_Y - m_Y, n_X - m_X + n_Y - m_Y)$ square matrix formed by all partial derivatives of $\boldsymbol{\Upsilon}^\epsilon$ with respect to $(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)$ at $\bar{\mathbf{a}} = (\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L); \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L))$, and $\mathcal{A}^\epsilon$ and $\mathcal{B}^\epsilon$ are matrices of dimension $(n_X - m_X + n_Y - m_Y, m_X + m_Y)$. The next lemma says that the solution to (15), if it exists, determines the followers' strategies.

LEMMA 1. *If* $\left|\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F)}}(\bar{\mathbf{a}})\right| \neq 0$, *then, for all* $\varepsilon > 0$, *there exist unique functions* $\sigma_i^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathcal{S}_i$, *with* $b_{i,\epsilon} = \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$, $i = m_X + 1, ..., n_X$, *and* $\varphi_j^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathcal{S}_j$, *with* $q_{j,\epsilon} = \varphi_j^\epsilon(\mathbf{q}^L; \mathbf{b}^L; \epsilon)$, $j = m_Y + 1, ..., n_Y$. *Moreover,* $\sigma_i^\epsilon(.) \in \mathcal{C}^2(\mathbf{S}, \mathcal{S}_i)$, $i = m_X + 1, ..., n_X$, *and* $\varphi_j^\epsilon(.) \in \mathcal{C}^2(\mathbf{S}, \mathcal{S}_j)$, $j = m_Y + 1, ..., n_Y$.

PROOF. See Appendix C.

Lemma 1 provides a sufficient condition for the existence and uniqueness of continuous differentiable strategies. If the Jacobian of $\boldsymbol{\Upsilon}^\epsilon(.)$ is a linear map which is invertible, i.e. a $\mathcal{C}^1$-diffeomorphism, then there exists a unique fixed point to (9) which consists of twice continuously differentiable strategies. The next proposition state that there the followers' strategies are bounded.

PROPOSITION 3. *Let* $\boldsymbol{\sigma}^\epsilon = (\sigma_{M_1+1}^\epsilon, ..., \sigma_{N_1}^\epsilon)$ *and* $\boldsymbol{\varphi}^\epsilon = (\varphi_{M_2+1}^\epsilon, ..., \varphi_{N_2}^\epsilon)$ *be respectively* $(n_X - m_X)$ *and* $(n_Y - m_Y)$ *dimensional vector functions. Consider* $\mathcal{J}_{\boldsymbol{\sigma}^\epsilon_{\mathbf{q}_\epsilon^L}}(\bar{\mathbf{a}}) = \left[\frac{\partial \boldsymbol{\sigma}^\epsilon(.)}{\partial \mathbf{q}_\epsilon^L}\right]$ *and* $\mathcal{J}_{\boldsymbol{\varphi}^\epsilon_{\mathbf{q}_\epsilon^L}}(\bar{\mathbf{a}}) = \left[\frac{\partial \boldsymbol{\varphi}^\epsilon(.)}{\partial \mathbf{q}_\epsilon^L}\right]$. *Then,* $\mathcal{J}_{\boldsymbol{\sigma}^\epsilon_{\mathbf{q}_\epsilon^L}}(\bar{\mathbf{a}}) \in [-\mathbf{I}, \mathbf{I})$ *and* $\mathcal{J}_{\boldsymbol{\varphi}^\epsilon_{\mathbf{q}_\epsilon^L}}(\bar{\mathbf{a}}) \geq \mathbf{0}$, *where the unit matrix* $\mathbf{I}$ *and the null matrix* $\mathbf{0}$ *are of dimension* $(n_X - m_X, m_X)$ *and* $(n_Y - m_Y, m_X)$ *respectively. In addition,* $\mathcal{J}_{\boldsymbol{\varphi}^\epsilon_{\mathbf{b}_\epsilon^L}}(\bar{\mathbf{a}}) \in [-\mathbf{I}, \mathbf{I})$ *and* $\mathcal{J}_{\boldsymbol{\sigma}^\epsilon_{\mathbf{b}_\epsilon^L}}(\bar{\mathbf{a}}) \geq \mathbf{0}$, *where* $\mathbf{I}$ *and* $\mathbf{0}$ *are of dimension* $(n_Y - m_Y, m_Y)$ *and* $(n_X - m_X, m_Y)$ *respectively.*

PROOF. See Appendix D.

Consider now the subgame $\boldsymbol{\Gamma}_L^\epsilon$. Define the two families of followers' strategies in Γ^ϵ as $\boldsymbol{\sigma}^\epsilon : \mathcal{S}^L \times \mathbb{R}_{++} \to \prod_{i=m_X+1}^{n_X} \mathcal{S}_i$, with $\mathbf{q}_\epsilon^F = \boldsymbol{\sigma}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$, and as $\boldsymbol{\varphi}^\epsilon$:

$\mathcal{S}^L \times \mathbb{R}_{++} \to \prod_{j=m_Y+1}^{n_Y} \mathcal{S}_j$, with $\mathbf{b}_\epsilon^F = \boldsymbol{\varphi}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)$. In particular, $\boldsymbol{\sigma}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \in \mathcal{C}^2$ and $\boldsymbol{\varphi}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \in \mathcal{C}^2$. By virtue of Lemma 1, the price function that maps leaders' strategies into a price and the reduced form payoffs of leaders are well-defined. Indeed, each leader knows how the market price is affected by the followers' reactions. Let the price function be $p_X^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \triangleq p_X(\mathbf{q}_\epsilon^L, \boldsymbol{\sigma}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon))$. As $\boldsymbol{\sigma}^\epsilon(.) \in \mathcal{C}^2$ and $\boldsymbol{\varphi}^\epsilon(.) \in \mathcal{C}^2$, then $p_X^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \in \mathcal{C}^2$. Thus, leaders' reduced form payoffs are $\pi_{i,\epsilon}(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, \boldsymbol{\sigma}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$, for each $i = 1, ..., m_X$, and $\pi_{j,\epsilon}(\mathbf{q}_\epsilon^L, \boldsymbol{\sigma}^\epsilon(\mathbf{q}_\epsilon^L; b_{j,\epsilon}, \mathbf{b}_{-j,\epsilon}^L; \epsilon); b_{j,\epsilon}, \mathbf{b}_{-j,\epsilon}^L, \boldsymbol{\varphi}^\epsilon(\mathbf{q}_\epsilon^L; b_{j,\epsilon}, \mathbf{b}_{-j,\epsilon}^L; \epsilon)$, for each $j = 1, ..., m_Y$. The next proposition relies to existence and continuity of leaders' best responses (see Definition 3).

PROPOSITION 4. *Let Assumption 2 be satisfied. Then, for all $\epsilon > 0$, the best responses* $\phi_i^\epsilon : \mathbf{S}_{-i}^L \times \prod_{j=1}^{m_Y} \mathcal{S}_j \times \mathbb{R}_{++} \to S_i$, *with* $(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) \mapsto \phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$, $i = 1, ..., m_X$, *and* $\psi_j^\epsilon : \prod_{i=1}^{m_X} \mathcal{S}_i \times \mathbf{S}_{-j}^L \times \mathbb{R}_{++} \to S_j$, *with* $(\mathbf{q}_\epsilon^L; \mathbf{b}_{-j,\epsilon}^L; \epsilon) \mapsto \psi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_{-j,\epsilon}^L; \epsilon)$, $j = 1, ..., m_Y$, *exist and are continuous functions.*

PROOF. See Appendix E.

We now turn to the existence of an ϵ-SNE. The next lemma shows that the optimal behavior of traders are mutually consistent in the entire perturbed game.

LEMMA 2 (Existence of ϵ-SNE). *Consider $\mathbf{\Gamma}^\epsilon$, and let Assumptions 1 and 2 be satisfied. Then, for all $\epsilon > 0$, there exists an ε-Stackelberg-Nash equilibrium of $\mathbf{\Gamma}^\epsilon$.*

PROOF. We show that the optimal strategic behavior are mutually consistent, i.e., there is a pure strategy SPNE for the entire perturbed game $\mathbf{\Gamma}^\varepsilon$, which constitutes a NE of each perturbed subgame $\mathbf{\Gamma}_L^\epsilon$ and $\mathbf{\Gamma}_F^\alpha$. We first show that $\mathbf{\Gamma}_L^\epsilon$ has a NE. To this end, let $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L) \in S^L$, and define the family of functions $\mathbf{\Lambda}_L^\epsilon : \mathcal{S}^L \times \mathbb{R}_{++} \to \mathcal{S}^L$, with $\mathbf{\Lambda}_L^\epsilon(\mathbf{q}_\epsilon^L, \boldsymbol{\sigma}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L)) = \times_{i=1}^{m_X} \phi_i \times_{j=1}^{m_Y} \psi_j$, where the functions ϕ_i^ϵ, $i = 1, ..., m_X$, and ψ_j^ϵ, $j = 1, ..., m_Y$, are well-defined from Proposition 4 (see Appendix E). The function $\mathbf{\Lambda}_L^\epsilon$ is a continuous function (as the product of continuous functions ϕ_i^ϵ, $i = 1, ..., m_X$, and ψ_j^ϵ, $j = 1, ..., m_Y$, from Proposition 4) over a compact and convex subset of Euclidean space (as the product of compact and convex sets S_i, $i = 1, ..., m_X$, and S_j, $j = 1, ..., m_Y$). Then, by the Brouwer Fixed Point Theorem, the function $\mathbf{\Lambda}_L^\epsilon$ admits a fixed point, namely $(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$, which is a NE of $\mathbf{\Gamma}_L^\epsilon$. Next, we show $\mathbf{\Gamma}_F^\epsilon$ has a NE. Define $\mathbf{\Lambda}_F^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to S^F$, with $\mathbf{\Lambda}_F^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F) = \times_{i=m_X+1}^{n_X} \phi_i^\epsilon \times_{j=m_Y+1}^{n_Y} \psi_j^\epsilon$, where the functions ϕ_i^ϵ, $i = m_X + 1, ..., n_X$, and ψ_j^ϵ, $j = m_Y + 1, ..., n_Y$, are known to exist from Proposition 1. Let $(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L) = (\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$. The function $\mathbf{\Lambda}_F^\epsilon(\tilde{\mathbf{q}}_\epsilon^L, \mathbf{q}_\epsilon^F; \tilde{\mathbf{b}}_\epsilon^L, \mathbf{b}_\epsilon^F)$ is continuous on $\mathbf{S}$, a compact and convex set of Euclidean space. Then, it has a fixed point, namely $(\tilde{\mathbf{q}}_\epsilon^F; \tilde{\mathbf{b}}_\epsilon^F)$, which is a NE of $\mathbf{\Gamma}_F^\epsilon$. Finally, from Lemma 1, for all $\epsilon > 0$, as $(\mathbf{q}_\epsilon^F; \mathbf{b}_\epsilon^F) = (\boldsymbol{\sigma}^\epsilon(\mathbf{q}_\epsilon^{\mathbf{L}}; \mathbf{b}_\epsilon^L; \epsilon); \boldsymbol{\varphi}^\epsilon(\mathbf{q}_\epsilon^{\mathbf{L}}; \mathbf{b}_\epsilon^L; \epsilon))$. If $(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L)$ is a fixed point, then, by using Lemma 1, and by continuity of $\boldsymbol{\sigma}^\epsilon(.)$ and $\boldsymbol{\varphi}^\epsilon(.)$, we deduce $(\tilde{\mathbf{q}}_\epsilon^F; \tilde{\mathbf{b}}_\epsilon^F) = (\boldsymbol{\sigma}^\epsilon(\tilde{\mathbf{q}}_\epsilon^{\mathbf{L}}; \tilde{\mathbf{b}}_\epsilon^L; \epsilon); \boldsymbol{\varphi}^\epsilon(\tilde{\mathbf{q}}_\epsilon^{\mathbf{L}}; \tilde{\mathbf{b}}_\epsilon^L; \epsilon))$ is a fixed point of $\mathbf{\Gamma}_F^\epsilon$, for all $\epsilon > 0$. Then, $(\tilde{\mathbf{q}}_\varepsilon^L, \tilde{\mathbf{q}}_\varepsilon^F; \tilde{\mathbf{b}}_\varepsilon^L, \tilde{\mathbf{b}}_\varepsilon^F)$ is a fixed point of $\mathbf{\Gamma}^\epsilon$.■

The next step consists of showing that the market price generated by an ϵ-SNE is strictly positive and finite. Indeed, the next lemma concerns the existence of uniform bounds on relative price in an ϵ-SNE.

LEMMA 3. *Assume there are at least one leader and one follower of each type, and let* $\tilde{p}_X^{\epsilon} = p_X^{\epsilon}(\tilde{\mathbf{q}}_{\epsilon}^L, \tilde{\mathbf{q}}_{\epsilon}^F(\tilde{\mathbf{q}}_{\epsilon}^L; \tilde{\mathbf{b}}_{\epsilon}^L); \tilde{\mathbf{b}}_{\epsilon}^L, \tilde{\mathbf{b}}_{\epsilon}^F(\tilde{\mathbf{q}}_{\epsilon}^L; \tilde{\mathbf{b}}_{\epsilon}^L))$. *Then, in an* ϵ*-SNE, there exist uniform bounds* $\xi_1 > 0$ *and* $\xi_2 > 0$ *such that*:

$$\forall \epsilon > 0,\ \xi_1 < \tilde{p}_X^{\epsilon} < \xi_2. \tag{16}$$

PROOF. See Appendix F.

It is worth noting that, unlike Dubey and Shubik (1978), the existence of such uniform bounds on price must hold in each perturbed subgame. In particular, we have to take into account that, in the perturbed subgame between leaders, the rational beliefs of leaders about the followers' reactions matter.

The next lemma, whose proof adapts to our setting the study of Cournot equilibria in Bloch and Ferrer (2001), is linked to the existence of an ε-SNE with trade.

LEMMA 4 (Existence of ϵ-SNE with trade). *Consider* $\mathbf{\Gamma}^{\epsilon}$, *and let Assumptions 1 and 2 be satisfied. Then, for all* $\epsilon > 0$, *there exists an* ε*-SNE with trade of* $\mathbf{\Gamma}^{\epsilon}$.

PROOF. We have to show that there are non trivial equilibrium strategies in each stage, i.e., there exist lower and upper uniform bounds on equilibrium supplies such that there are at least one leader and one follower of the first type (resp. second type) for whom $0 < \tilde{q}_{i,\epsilon} < \alpha_i$ (resp. $0 < \tilde{b}_{j,\epsilon} < \beta_j$).

Follower i. Consider the payoff given by (12). Let $\sigma_i^{\epsilon}(\mathbf{q}_{\epsilon}^L; \mathbf{b}_{\epsilon}^L; \epsilon) \in \mathcal{S}_i$. We have to show that there are $\underline{q}_i, \bar{q}_i \in S_i$ such that $0 < \underline{q}_i \leqslant \sigma_i^{\epsilon}(\tilde{\mathbf{q}}_{\epsilon}^L; \tilde{\mathbf{b}}_{\epsilon}^L; \epsilon) \leqslant \bar{q}_i < \alpha_i$, for at least one i, $i = m_X+1, ..., n_X$. Fix the strategies of all other traders in equilibrium. Follower i's marginal payoff may be written (see (A2) in Appendix A):

$$\frac{\partial \pi_i^{\epsilon}}{\partial q_{i,\epsilon}} = -\frac{\partial u_i}{\partial x_i} + p_X^{\epsilon} \frac{Q_{-i,\epsilon} + \varepsilon}{q_{i,\epsilon} + Q_{-i,\epsilon} + \varepsilon} \frac{\partial u_i}{\partial y_i}, \text{ for all } \epsilon > 0. \tag{17}$$

From Proposition 1, there exists $\phi_i^{\epsilon}(\mathbf{q}_{\varepsilon}^L, \mathbf{q}_{-i,\varepsilon}^F; \mathbf{b}_{\epsilon}^L, \mathbf{b}_{\epsilon}^F; \epsilon) \geqslant 0$, $i = m_X+1, ..., n_X$. In addition, from Lemma 1, there exists $q_{i,\epsilon} = \sigma_i^{\epsilon}(\mathbf{q}_{\epsilon}^L; \mathbf{b}_{\epsilon}^L; \epsilon) \geqslant 0$, $i = m_X+1, ..., n_X$. Then, in equilibrium we have $\tilde{q}_{i,\epsilon} = \sigma_i^{\epsilon}(\tilde{\mathbf{q}}_{\epsilon}^L; \tilde{\mathbf{b}}_{\epsilon}^L; \epsilon) \geqslant 0$, $i = m_X + 1, ..., n_X$. Let $MRS_{X,Y}^i(\tilde{\mathbf{z}}_{i,\epsilon}) = \frac{\partial u_i / \partial x_i}{\partial u_i / \partial y_i}_{|\mathbf{z}_{i,\epsilon} = \tilde{\mathbf{z}}_{i,\epsilon}}$, so (17) may be written:

$$\frac{\partial \pi_i^{\epsilon}}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon} = \tilde{q}_{i,\epsilon}} = \frac{\partial u_i}{\partial y_i}(\tilde{p}_X^{\epsilon} - MRS_{X,Y}^i(\tilde{\mathbf{z}}_{i,\epsilon})), \text{ for all } \epsilon > 0. \tag{18}$$

Consider the case $\sigma_i^{\epsilon}(\tilde{\mathbf{q}}_{\epsilon}^L; \tilde{\mathbf{b}}_{\epsilon}^L; \epsilon) \geqslant \underline{b}_i > 0$. As $\frac{Q_{-i,\epsilon} + \varepsilon}{q_{i,\epsilon} + Q_{-i,\epsilon} + \varepsilon} \leqslant 1$ and, from Lemma 3, we have $\tilde{p}_X^{\epsilon} > \xi_1$, then (18) may be written:

$$\frac{\partial \pi_i^{\epsilon}}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon} = \tilde{q}_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i}(\xi_1 - MRS_{X,Y}^i(\tilde{\mathbf{z}}_{i,\epsilon})), \text{ for all } \epsilon > 0. \tag{19}$$

From (2a)-(2c), we deduce $\frac{\partial MRS_{X,Y}^i}{\partial q_{i,\epsilon}} > 0$. Assume $\sigma_i^{\epsilon}(.; \epsilon) = 0$. Then, from (2d), $\lim_{q_{i,\epsilon} \to 0} MRS_{X,Y}^i = 0$, so (19) becomes $\frac{\partial \pi_i^{\epsilon}}{\partial q_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i}\xi_1$. But, from (2d), we have $\lim_{q_{i,\epsilon} \to 0} \frac{\partial u_i}{\partial y_i} = \lim_{y_i \to 0} \frac{\partial u_i}{\partial y_i} = \infty$, so we deduce $\frac{\partial \pi_i^{\epsilon}}{\partial q_{i,\epsilon}} > \infty$. A contradiction.

Therefore, there must be $\underline{q}_i > 0$, with $\underline{q}_i = \sigma_i^\epsilon(\underline{q}_\epsilon^L; \underline{b}_\epsilon^L; \epsilon)$ and $\underline{q}_i \in \mathcal{S}_i$, such that $\frac{\partial MRS_{X,Y}^i}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon}=\underline{q}_i} = \xi_1$. As $\frac{\partial \pi_{i,\epsilon}}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon}=\underline{q}_i} > 0$, then for all $\sigma_i^\epsilon(.;\epsilon) \in \mathcal{S}_i$, we have $\sigma_i^\epsilon(.;\epsilon) \geqslant \underline{q}_i > 0$. Then, $\sigma_i^\epsilon(.;\epsilon) > 0$, so $\lambda_{i,\epsilon} = 0$ in (A6), for at least one $i \in \{1,...,m_X\}$. Likewise, $0 < \underline{b}_j \leqslant \varphi_j^\epsilon .;\epsilon)$, $j = m_Y + 1, ..., n_Y$.

Consider now the case $q_{i,\epsilon} \leqslant \bar{q}_i < \alpha_i$. As $p_X^\epsilon < \xi_2$ and $\frac{Q_{-i,\epsilon}+\varepsilon}{q_{i,\epsilon}+Q_{-i,\epsilon}+\varepsilon} \leqslant 1$, then:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon}=\bar{q}_{i,\epsilon}} < \frac{\partial u_i}{\partial y_i}(\xi_2 - MRS_{X,Y}^i(\tilde{\mathbf{z}}_{i,\epsilon})), \text{ for all } \epsilon > 0. \tag{20}$$

From (2a)-(2c), $\frac{\partial MRS_{X,Y}^i}{\partial q_{i,\epsilon}} > 0$. In addition, from (2d), $\lim_{q_{i,\epsilon}\to 0} MRS_{X,Y}^i = 0$ and $\lim_{q_{i,\epsilon}\to \alpha_i} MRS_{X,Y}^i = \infty$. Then, there is $\bar{q}_i < \alpha_i$, with $\bar{q}_i = \bar{\sigma}_i^\epsilon(.;\epsilon)$ and $\bar{\sigma}_i^\epsilon(.;\epsilon) \in \mathcal{S}_i$, such that $(\frac{\partial MRS_{X,Y}^i}{\partial q_{i,\epsilon}})_{|q_{i,\epsilon}=\bar{q}_i} = \xi_2$. Then, from (20), $(\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}})_{|q_{i,\epsilon}=\bar{q}_i} < 0$, where π_i^ϵ is strictly concave in $q_{i,\epsilon}$ on $[0, \alpha_i]$. Then, for all $\tilde{q}_{i,\varepsilon} \in \mathcal{S}_i$, we get $\tilde{q}_{i,\varepsilon} \leqslant \bar{q}_i$, so $\lambda_{i,\epsilon} = 0$ in (A6). But, then, $\varphi_i^\epsilon(.;\epsilon) \leqslant \bar{q}_i < \alpha_i$ for at least one follower i.

Leader i. Fix the strategies of all other leaders in equilibrium. As, in the first stage, $\tilde{p}_X^\epsilon = p_X^\epsilon(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L; \epsilon)$, where $p_X^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon) \triangleq p_X(\mathbf{q}_\epsilon^L, \boldsymbol{\sigma}^\epsilon(.;\epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(.;\epsilon))$, leader i's marginal payoff may be written:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon}=\tilde{q}_{i,\epsilon}} = \frac{\partial u_i}{\partial y_i}(\chi \tilde{p}_X^\epsilon - MRS_{X,Y}^i(\tilde{\mathbf{z}}_{i,\epsilon})), \text{ for all } \epsilon > 0. \tag{21}$$

where $\chi \equiv 1 - (1 + \nu_\epsilon^X)\frac{\tilde{q}_{i,\epsilon}}{\tilde{q}_{i,\epsilon}+\tilde{Q}_{-i,\epsilon}+\epsilon} + \eta_\epsilon^X \frac{\tilde{q}_{i,\epsilon}}{\tilde{B}_\epsilon+\epsilon}$, with $\chi \in [0, 1]$, is the inverse of the markup (see (E2) in Appendix E).

Consider the case $\tilde{q}_{i,\varepsilon} \geqslant \underline{q}_i > 0$. As $\tilde{p}_X^\epsilon > \xi_1$ and $\chi \leqslant 1$, then (21) is $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon}=\tilde{q}_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i}(\xi_1 - MRS_{X,Y}^i(\tilde{\mathbf{z}}_{i,\epsilon}))$, for all $\epsilon > 0$. From (2a)-(2c), $\frac{\partial MRS_{X,Y}^i}{\partial q_{i,\epsilon}} > 0$. Assume $\tilde{q}_{i,\epsilon} = 0$. Then, $\chi = 1$, and, from (2d), $\lim_{q_{i,\epsilon}\to 0} MRS_{X,Y}^i = 0$, so (21) is $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \frac{\partial u_i}{\partial y_i}\xi_1$. But, from (2d), $\lim_{q_{i,\epsilon}\to 0} \frac{\partial u_i}{\partial y_i} = \lim_{y_i \to 0} \frac{\partial u_i}{\partial y_i} = \infty$, so $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} > \infty$. A contradiction. Therefore, there is $\underline{q}_i > 0$, with $\underline{q}_i \in \mathcal{S}_i$, such that $\frac{\partial MRS_{X,Y}^i}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon}=\underline{q}_i} = \xi_1$. As $\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}}_{|q_{i,\epsilon}=\underline{q}_i} > 0$, then for all $\tilde{q}_{i,\epsilon} \in \mathcal{S}_i$, $\tilde{q}_{i,\epsilon} \geqslant \underline{q}_i > 0$. Then, $\tilde{q}_{i,\epsilon} > 0$, so $\mu_{i,\epsilon} = 0$ in (E2), for at least one $i \in \{1, ..., m_X\}$.

The proof of $0 < \underline{b}_j \leqslant b_j \leqslant \tilde{b}_{j,\epsilon} < \beta_j$ for at least one $j \in T_Y$, follows the same steps as the one provided for type 1 traders.■

Finally, we show the SNE is an equilibrium point (EP), which we now define.

DEFINITION 6. A SNE $(\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L); \tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L))$ is an *equilibrium point* of $\mathbf{\Gamma}$ if there exist sequences $\{\epsilon_n\}_{n=1}^\infty$ and $\{(\tilde{\mathbf{q}}_{\epsilon_n}^L, \tilde{\mathbf{q}}_{\epsilon_n}^F(\tilde{\mathbf{q}}_{\epsilon_n}^L; \tilde{\mathbf{b}}_{\epsilon_n}^L); \tilde{\mathbf{b}}_{\epsilon_n}^L, \tilde{\mathbf{b}}_{\epsilon_n}^F(\tilde{\mathbf{q}}_{\epsilon_n}^L; \tilde{\mathbf{b}}_{\epsilon_n}^L))\}_{n=1}^\infty$ such that:

1. $\epsilon_n > 0$ and $\lim_{n\to\infty}\{\epsilon_n\} = 0$;
2. $(\tilde{\mathbf{q}}_{\epsilon_n}^L, \tilde{\mathbf{q}}_{\epsilon_n}^F(\tilde{\mathbf{q}}_{\epsilon_n}^L; \tilde{\mathbf{b}}_{\epsilon_n}^L); \tilde{\mathbf{b}}_{\epsilon_n}^L, \tilde{\mathbf{b}}_{\epsilon_n}^F(\tilde{\mathbf{q}}_{\epsilon_n}^L; \tilde{\mathbf{b}}_{\epsilon_n}^L))$ is a Nash equilibrium of $\mathbf{\Gamma}^{\epsilon_n}$;
3. $\lim_{n\to\infty}\{(\tilde{\mathbf{q}}_{\epsilon_n}^L, \tilde{\mathbf{q}}_{\epsilon_n}^F(\tilde{\mathbf{q}}_{\epsilon_n}^L; \tilde{\mathbf{b}}_{\epsilon_n}^L); \tilde{\mathbf{b}}_{\epsilon_n}^L, \tilde{\mathbf{b}}_{\epsilon_n}^F(\tilde{\mathbf{q}}_{\epsilon_n}^L; \tilde{\mathbf{b}}_{\epsilon_n}^L))\} = (\tilde{\mathbf{q}}^L, \tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L); \tilde{\mathbf{b}}^L, \tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L; \tilde{\mathbf{b}}^L))$.

LEMMA 5 (SNE is an EP). *Consider the game $\mathbf{\Gamma}$, and let Assumptions 1 and 2 be satisfied. Then, the SNE with trade is an equilibrium point of $\mathbf{\Gamma}$.*

PROOF. Consider a sequence $\{\epsilon_n\}$ such that $\lim_{n\to\infty}\{\epsilon_n\}=0$. Pick one sequence of strategies $\{(\mathbf{q}^L_{\epsilon_n},\mathbf{q}^F_{\epsilon_n}(\mathbf{q}^L_{\epsilon_n};\mathbf{b}^L_{\epsilon_n});\mathbf{b}^L_{\epsilon_n},\mathbf{b}^F_{\epsilon_n}(\mathbf{q}^L_{\epsilon_n};\mathbf{b}^L_{\epsilon_n}))\}$, $n\in\{1,2,...\}$. Consider the subgame $\mathbf{\Gamma}^{\epsilon_n}_F$. From Lemma 1, there exist $q_{i,\epsilon}:=\sigma^{\epsilon}_i(\mathbf{q}^L_{\epsilon};\mathbf{b}^L_{\epsilon};\epsilon)$, for each $i=m_X+1,...,n_X$, and $b_{j,\epsilon}:=\varphi^{\epsilon}_j(\mathbf{q}^L_{\epsilon};\mathbf{b}^L_{\epsilon};\epsilon)$, for each $j=m_Y+1,...,n_Y$. Consider, for each $i=m_X+1,...,n_X$, and for each $j=m_Y+1,...,n_Y$, the sequence of strategies $\{\sigma^{\epsilon_n}_i(\mathbf{q}^L_{\epsilon_n};\mathbf{b}^L_{\epsilon_n};\epsilon_n);\varphi^{\epsilon_n}_j(\mathbf{q}^L_{\epsilon_n};\mathbf{b}^L_{\epsilon_n};\epsilon_n)\}$, $n=1,2,...$, which are defined over compact sets. Let $(\mathbf{q}^L_{\epsilon_{k_n}};\mathbf{b}^L_{\epsilon_{k_n}};\epsilon_{k_n})$ be a leaders' strategy profile of the subgame $\mathbf{\Gamma}^{\epsilon_{k_n}}_L$. Then, for each $i=m_X+1,...,n_X$, and each $j=m_Y+1,...,n_Y$, there is a subsequence $\{\sigma^{\epsilon_{k_n}}_i(\mathbf{q}^L_{\epsilon_{k_n}};\mathbf{b}^L_{\epsilon_{k_n}};\epsilon_{k_n}),\varphi^{\epsilon_{k_n}}_j(\mathbf{q}^L_{\epsilon_{k_n}};\mathbf{b}^L_{\epsilon_{k_n}};\epsilon_{k_n})\}$ such that $\lim\limits_{n\to\infty}\{\sigma^{\epsilon_{k_n}}_i(\mathbf{q}^L_{\epsilon_{k_n}};\mathbf{b}^L_{\epsilon_{k_n}};\epsilon_{k_n});\varphi^{\epsilon_{k_n}}_j(\mathbf{q}^L_{\epsilon_{k_n}};\mathbf{b}^L_{\epsilon_{k_n}};\epsilon_{k_n})\}=\{\sigma_i(\mathbf{q}^L;\mathbf{b}^L);\varphi_j(\mathbf{q}^L;\mathbf{b}^L)\}$ as $\lim\limits_{n\to\infty}\{\sigma^{\epsilon_{k_n}}_i(.;\epsilon_{k_n});\varphi^{\epsilon_{k_n}}_j(.;\epsilon_{k_n})\}=\{\sigma_i(.);\varphi_j(.)\}$, $i=m_X+1,...,n_X$, $j=m_Y+1,...,n_Y$. But $(q_i;b_j):=(\sigma_i(\mathbf{q}^L;\mathbf{b}^L);\varphi_j(\mathbf{q}^L;\mathbf{b}^L)$, $i=m_X+1,...,n_X$, $j=m_Y+1,...,n_Y$. In addition, from Lemma 4, we have $\underline{q}_i\leqslant\tilde{q}_i\leqslant\bar{q}_i$, $i=m_X+1,...,n_X$, and $\underline{b}_j\leqslant\tilde{b}_j\leqslant\bar{b}_j$, $j=m_Y+1,...,n_Y$. By continuity of the payoff functions of the followers (see Appendix A), we deduce $(q_i;b_j):=(\sigma_i(\mathbf{q}^L;\mathbf{b}^L);\varphi_j(\mathbf{q}^L;\mathbf{b}^L)$ is well-defined, for each $i=m_X+1,...,n_X$, and for each $j=m_Y+1,...,n_Y$, so $(\mathbf{q}^F;\mathbf{b}^F):=(\boldsymbol{\sigma}(\mathbf{q}^L;\mathbf{b}^L);\boldsymbol{\varphi}(\mathbf{q}^L;\mathbf{b}^L))$ is a well-defined strategy profile of $\mathbf{\Gamma}_F$. Consider now the subgame $\mathbf{\Gamma}^{\epsilon_n}_L$. From Lemma 4, we know that there exists an ε-SNE with trade of $\Gamma^{\epsilon_n}_L$, i.e. there is a strategy profile $\{(\tilde{\mathbf{q}}^L_{\epsilon_n};\tilde{\mathbf{b}}^L_{\epsilon_n})\}$, for which, for at least one leader of each type, we have that $\underline{q}_i\leqslant\tilde{q}_{i,\epsilon_n}\leqslant\bar{q}_i$, $i=1,...,m_X$, and $\underline{b}_j\leqslant\tilde{b}_{j,\epsilon_n}\leqslant\bar{b}_j$, $j=1,...,m_Y$, for $n=1,2,...$. Thus, the sequence $\{(\tilde{q}_{i,\epsilon_n};\tilde{b}_{j,\epsilon_n})\}$, $i=m_X+1,...,n_X$, $j=m_Y+1,...,n_Y$, is defined over a compact set. Then, from the Bolzano-Weierstrass Theorem (see Corollary 4.7, p. 25 in Aliprantis et al. 1998), there exists, for each $i=1,...,m_X$, $j=1,...,m_Y$, a subsequence $\{(\tilde{q}_{i,\epsilon_{k_n}};\tilde{b}_{j,\epsilon_{k_n}})\}^{\infty}_{n=1}$ which converges to a limit point $(\tilde{q}_j;\tilde{b}_i)$, where $\underline{q}_i\leqslant\tilde{q}_i\leqslant\bar{q}_i$, for each $i=1,...,m_X$, and $\underline{b}_j\leqslant\tilde{b}_j\leqslant\bar{b}_j$, for each $j=1,...,m_Y$, from Lemma 4. As the payoff functions of the leaders are strictly concave (see Appendix D), they are continuous, so $(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L)=(\tilde{q}_1,...,\tilde{q}_{m_X};\tilde{b}_1,...,\tilde{b}_{m_Y})$ is an EP of $\mathbf{\Gamma}_L$. As $(\tilde{\mathbf{q}}^F;\tilde{\mathbf{b}}^F):=(\boldsymbol{\sigma}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\boldsymbol{\varphi}(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L))$, which is an EP of $\mathbf{\Gamma}_F$. But then, $(\tilde{\mathbf{q}}^L,\tilde{\mathbf{q}}^F;\tilde{\mathbf{b}}^L,\tilde{\mathbf{b}}^F)$ is an interior pure strategy SPNE of $\mathbf{\Gamma}$. Then, the SNE with trade is an EP of $\mathbf{\Gamma}$, which means there exists a strategy profile $(\tilde{\mathbf{q}}^L,\tilde{\mathbf{q}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L);\tilde{\mathbf{b}}^L,\tilde{\mathbf{b}}^F(\tilde{\mathbf{q}}^L;\tilde{\mathbf{b}}^L))$, which is a non autarkic SNE of $\mathbf{\Gamma}$.■

4. DISCUSSION

The following examples deserve three purposes. First, they illustrate that our main result captures new insights on competition in bilateral oligopolies, and, thereby, each of them puts forward the main differences with the outcome of the Cournot-Nash game. Second, they buttress the logic of our approach. Third, they test the robustness of Assumption 2, i.e. the differentiability, the strict quasi-concavity and the behavior of the indifference curves along the boundary of the consumption sets. Example 1 computes a SNE when Assumption 2 is satisfied. Example 2 shows Assumption 2c is not necessary. Example 3 illustrates existence failure. Example 4 shows a SNE may exist even if Assumptions 2a, 2c and 2d do not hold for some traders. In each case, we also compute the Cournot-Nash equilibrium (CNE) supplies and the competitive equilibrium (CE) supplies. In all examples Assumption 1 is $\alpha_i=1$, for all $i\in T_X$, and $\beta_j=1$, for all $j\in T_Y$.

4.1. A SNE under Assumption 2

Let $T_X = \{1, 2, 3, 4\}$ and $T_Y = \{1, 2, 3, 4\}$, with two leaders and two followers of each type, with:

$$u_k(x_k, y_k) = x_k.y_k,\ k = i, j,\ i, j = 1, ..., 4. \tag{22}$$

The CE supplies are given by $(q_1^*, q_2^*, q_3^*, q_4^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(b_1^*, b_2^*, b_3^*, b_4^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. In addition, the CNE supplies are given by $(\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Let us now compute the SNE. In the second stage of the game, for all strategy profiles $(q_1, q_2; b_1, b_2) \in \mathcal{S}^L$, the problems of both types of followers may be written:

$$\max_{\phi_i(\mathbf{q}_{-i};\mathbf{b})} (1 - q_i) \left(\frac{\sum_{j=1}^4 b_j}{q_i + \sum_{-i,-i\neq i} q_{-i}} q_i \right),\ i = 3, 4, \tag{23}$$

$$\max_{\psi_j(\mathbf{q};\mathbf{b}_{-j})} \left(\frac{\sum_{i=1}^4 q_i}{b_j + \sum_{-j,-j\neq j} b_{-j}} b_j \right) (1 - b_j),\ j = 3, 4. \tag{24}$$

As all followers of the same type must adopt the same strategy at equilibrium, the sufficient first-order conditions lead to the followers' best responses, which are given by:

$$\phi_3(q_1, q_2, q_4; \mathbf{b}) = -(q_1 + q_2 + q_4) + \sqrt{(q_1 + q_2 + q_4)^2 + (q_1 + q_2 + q_4)}, \tag{25}$$

$$\phi_4(q_1, q_2, q_3; \mathbf{b}) = -(q_1 + q_2 + q_3) + \sqrt{(q_1 + q_2 + q_3)^2 + (q_1 + q_2 + q_3)}, \tag{26}$$

$$\psi_3(\mathbf{q}; b_1, b_2, b_4) = -(b_1 + b_2 + b_4) + \sqrt{(b_1 + b_2 + b_4)^2 + (b_1 + b_2 + b_4)}, \tag{27}$$

$$\psi_4(\mathbf{q}; b_1, b_2, b_3) = -(b_1 + b_2 + b_3) + \sqrt{(b_1 + b_2 + b_3)^2 + (b_1 + b_2 + b_3)}. \tag{28}$$

To determine the followers' strategies, let $\boldsymbol{\Upsilon}(\Phi_3(.), \Phi_4(.); \Psi_3(.), \Psi_4(.)) = \mathbf{0}$, where $\Phi_i(\mathbf{q}_{-i}; \mathbf{b}) := q_i - \phi_i(\mathbf{q}_{-i}; \mathbf{b})$, $i = 3, 4$, and $\Psi_3(\mathbf{q}; \mathbf{b}_{-j}) := b_j - \psi_j(\mathbf{q}; \mathbf{b}_{-j})$, $j = 3, 4$. The Jacobian corresponding to (15) is given by:

$$\mathcal{J}_{\boldsymbol{\Upsilon}_{(\mathbf{q}^F;\mathbf{b}^F)}} = \begin{bmatrix} 1 & g & 0 & 0 \\ h & 1 & 0 & 0 \\ 0 & 0 & 1 & g' \\ 0 & 0 & h' & 1 \end{bmatrix}, \tag{29}$$

where $g \equiv 1 - \frac{q_1+q_3+\frac{1}{2}}{\sqrt{(q_1+q_3)^2+q_1+q_3}}$, $h \equiv 1 - \frac{q_1+q_2+\frac{1}{2}}{\sqrt{(q_1+q_2)^2+q_1+q_2}}$, $g' \equiv 1 - \frac{b_1+b_3+\frac{1}{2}}{\sqrt{(b_1+b_3)^2+b_1+b_3}}$ and $h' \equiv 1 - \frac{b_1+b_2+\frac{1}{2}}{\sqrt{(b_1+b_2)^2+b_1+b_2}}$. We get $\left|\mathcal{J}_{\boldsymbol{\Upsilon}_{(\mathbf{q}^\mathbf{F};\mathbf{b}^F)}}\right| = (1-gh)(1-g'h') \neq 0$ as $gh \neq 1$ and $g'h' \neq 1$, so $[\boldsymbol{\Upsilon}(\mathbf{0}]^{-1}$ is well defined. Then, Lemma 1 holds, and the followers' strategies are given by:

$$\sigma_i(q_1,q_2;b_1,b_2)=\frac{1}{6}-\frac{1}{3}(q_1+q_2)+\sqrt{\frac{1}{3}(q_1+q_2)^2+2(q_1+q_2)+\frac{1}{4}},\ i=3,4, \tag{30}$$

$$\varphi_j(q_1,q_2;b_1,b_2)=\frac{1}{6}-\frac{1}{3}(b_1+b_2)+\frac{1}{3}\sqrt{(b_1+b_2)^2+2(b_1+b_2)+\frac{1}{4}},\ j=3,4. \tag{31}$$

In the first stage, the problem of any leader consists of maximizing her reduced form payoff, with $p_X(q_1,q_2;b_1,b_2)=\frac{b_1+b_2+\varphi_3(b_1,b_2)+\varphi_4(b_1,b_2)}{q_1+q_2+\sigma_3(q_1,q_2)+\sigma_4(q_1,q_2)}$. Therefore, by using (30)-(31), the leaders' problems may be written:

$$\max_{\phi_i(q_{-i};b_1,b_2)}(1-q_i)\left(\frac{\frac{1}{3}+\frac{2}{3}(b_1+b_2)+\frac{2}{3}\sqrt{(b_1+b_2)^2+2(b_1+b_2)+\frac{1}{4}}}{\frac{1}{3}+\frac{2}{3}(q_1+q_2)+\frac{2}{3}\sqrt{(q_1+q_2)^2+2(q_1+q_2)+\frac{1}{4}}}\right)q_i,\ i=1,2, \tag{32}$$

$$\max_{\varphi_j(q_1,q_2;b_{-i})}\left(\frac{\frac{1}{3}+\frac{2}{3}(q_1+q_2)+\frac{2}{3}\sqrt{(q_1+q_2)^2+2(q_1+q_2)+\frac{1}{4}}}{\frac{1}{3}+\frac{2}{3}(b_1+b_2)+\frac{2}{3}\sqrt{(b_1+b_2)^2+2(b_1+b_2)+\frac{1}{4}}}\right)b_j(1-b_j),\ j=1,2. \tag{33}$$

Then, after some tedious computations, as all leaders of the same type must adopt the same strategy at equilibrium, the first-order conditions associated with problems (32)-(33) yield the unique solution $\tilde{q}_i=\tilde{b}_j=0.421907$, $i=1,2$, $j=1,2$. From (30)-(31), we deduce $(\tilde{q}_3,\tilde{q}_4)=(\tilde{b}_3,\tilde{b}_4)=(0.427986,0.427986)$.

Therefore, the SNE supplies are given by the pure strategy profiles:

$$(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q}_4)=(0.421907,0.421907,0.427986,0.427986), \tag{34}$$

$$(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3,\tilde{b}_4)=(0.421907,0.421907,0.427986,0.427986). \tag{35}$$

It is worth noting that the existence of the four strategies (30)-(31) as well as the SNE depends entirely on the fact that (29) has a nonzero determinant.

4.2. The boundary conditions

Let $T_X=\{1,2\}$ and $T_Y=\{1,2\}$, with:

$$u_i(x_i,y_i)=\gamma_i x_i+y_i,\ \gamma_i\in(0,1),\ i=1,2, \tag{36}$$

$$u_j(x_j,y_j)=x_j.y_j,\ j=1,2. \tag{37}$$

The CE supplies are given by $(q_1^*,q_2^*)=(\frac{1}{2\gamma},\frac{1}{2\gamma})$ and $(b_1^*,b_2^*)=(\frac{1}{2},\frac{1}{2})$. In addition, if $\gamma_1\neq\gamma_2$, the CNE supplies are $(\hat{q}_1,\hat{q}_2)=(\frac{2}{3}\frac{\gamma_2}{(\gamma_1+\gamma_2)^2},\frac{2}{3}\frac{\gamma_1}{(\gamma_1+\gamma_2)^2})$ and $(\hat{b}_1,\hat{b}_2)=(\frac{1}{3},\frac{1}{3})$, while if $\gamma_1=\gamma_2$, then $(\hat{q}_1,\hat{q}_2)=(\frac{1}{6\gamma},\frac{1}{6\gamma})$ and $(\hat{b}_1,\hat{b}_2)=(\frac{1}{3},\frac{1}{3})$.

Let us now compute the SNE. In the second stage of the game, the problems of both types of followers may be written:

$$\max_{\phi_2(q_1;b_1,b_2)} \quad \gamma_2(1-q_2) + \left(\frac{b_1+b_2}{q_1+q_2}q_2\right), \tag{38}$$

$$\max_{\psi_2(q_1,q_2;b_1)} \quad \left(\frac{q_1+q_2}{b_1+b_2}b_2\right).(1-b_2). \tag{39}$$

Then, the followers' best responses are given by:

$$\phi_2(q_1;b_1,b_2) = -q_1 + \sqrt{\frac{b_1+b_2}{\gamma_2}q_1}, \tag{40}$$

$$\psi_2(q_1,q_2;b_1) = -b_1 + \sqrt{(b_1)^2 + b_1}. \tag{41}$$

We now determine the followers' strategies. To this end, let $\Phi_2(q_1,q_2;b_1,b_2) := q_1 + q_2 - \sqrt{\frac{b_1+b_2}{\gamma_2}q_1}$ and $\Psi_2(q_1,q_2;b_1) := b_1 + b_2 - \sqrt{(b_1)^2+b_1}$. The Jacobian is given by:

$$\mathcal{J}_{\Upsilon_{(\mathbf{q}^F;\mathbf{b}^F)}} = \begin{bmatrix} 1 & -\frac{1}{2}\sqrt{\frac{q^1}{\gamma(b_1+b_2)}} \\ 0 & 1 \end{bmatrix}. \tag{42}$$

We have $\left|\mathcal{J}_{\Upsilon_{(\mathbf{q}^F;\mathbf{b}^F)}}\right| = 1$. Then, the followers' strategies exist and are given by:

$$\sigma_2(q_1;b_1) = -q_1 + \sqrt{\frac{1}{\gamma_2}\sqrt{(b_1)^2+b_1}q_1}, \tag{43}$$

$$\varphi_2(q_1;b_1) = -b_1 + \sqrt{(b_1)^2+b_1}. \tag{44}$$

In the first stage, any leader maximizes her reduced form payoff, with $p_X(q_1;b_1) = \frac{b_1+\varphi_2(q_1;b_1)}{q_1+\sigma_2(q_1;b_1)} = \sqrt{\frac{\gamma_2\sqrt{(b_1)^2+b_1}}{q_1}}$, so the problems of the two leaders may be written:

$$\max_{\{q_1\}} \gamma_1(1-q_1) + \sqrt{\gamma_2\sqrt{(b_1)^2+b_1}q_1}, \tag{45}$$

$$\max_{\{b_1\}} \sqrt{\frac{q_1}{\gamma_2\sqrt{(b_1)^2+b_1}}}b_1.(1-b_1). \tag{46}$$

Then, after some tedious computations, the first-order conditions associated with problems (45)-(46) yield the SNE supplies:

$$(\tilde{q}_1,\tilde{q}_2) = \left(\frac{\sqrt{2\sqrt{97}+62}}{48}\frac{\gamma_2}{(\gamma_1)^2}, \frac{\sqrt{2\sqrt{97}+62}}{24}\frac{1}{\gamma_1}\left(1-\frac{1}{2}\frac{\gamma_2}{\gamma_1}\right)\right), \tag{47}$$

$$(\tilde{b}_1,\tilde{b}_2) = \left(\frac{\sqrt{97}-5}{12}, \frac{5-\sqrt{97}+\sqrt{2\sqrt{97}+62}}{12}\right). \tag{48}$$

Therefore, there is a SNE with trade even if some traders have linear preferences. At least one trader (a leader and a follower of type 2) never makes a null demand for her "own" commodity: their indifference curves do not intersect the quantity axis. In addition, it can be checked that if leaders had linear utility functions, while followers had Cobb-Douglas utility functions, then there would be a SNE with trade. But if all traders had the same linear utility function, then the SNE would coincide with the CNE, which is autarkic (Cordella and Gabszewicz 1998).

4.3. Existence failure of SNE

Let $T_X = \{1, 2\}$ and $T_Y = \{1, 2\}$, with:

$$u_i(x_i, y_i) = \min\left(x_i - 1, \sqrt{(y_i)^2}\right), \ i = 1, 2, \tag{49}$$

$$u_j(x_j, y_j) = \min(\sqrt{(x_i)^2}, y_j - 1), \ j = 1, 2. \tag{50}$$

The CE supplies are given by $(q_1^*, q_2^*) = (0, 0)$ and $(b_1^*, b_2^*) = (0, 0)$ (autarky is Pareto optimal). The CNE supplies are $(\hat{q}_1, \hat{q}_2) = (0, 0)$ and $(\hat{b}_1, \hat{b}_2) = (0, 0)$.

Let us now compute the SNE. In the second stage of the game, the problems of both types of followers may be written:

$$\max_{\phi_2(q_1;b_1,b_2)} \ \min\left(-q_2, \sqrt{\left(\frac{b_1 + b_2}{q_1 + q_2} q_2\right)^2}\right), \tag{51}$$

$$\max_{\psi_2(q_1,q_2;b_1)} \ \min\left(\sqrt{\left(\frac{q_1 + q_2}{b_1 + b_2} b_2\right)^2}, -b_2\right). \tag{52}$$

The followers' best responses are given by:

$$\phi_2(q_1; b_1, b_2) = -q_1 + (b_1 + b_2), \tag{53}$$

$$\psi_2(q_1, q_2; b_1) = -b_1 + (q_1 + q_2). \tag{54}$$

Let $\mathbf{\Upsilon}(\Phi_2(.); \Psi_2(.)) = \mathbf{0}$, where $\Phi_2(q_1, q_2; b_1, b_2) := q_1 + q_2 - (b_1 + b_2)$ and $(q_1, q_2; b_1, b_2) := b_1 + b_2 - (q_1 + q_2)$. The Jacobian is given by:

$$\mathcal{J}_{\mathbf{\Upsilon}_{(\mathbf{q}^F;\mathbf{b}^F)}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{55}$$

As $\left|\mathcal{J}_{\mathbf{\Upsilon}_{(\mathbf{q}^F;\mathbf{b}^F)}}\right| = 0$, $[\mathbf{\Upsilon}(\mathbf{0}]^{-1} = \{\varnothing\}$, so $\mathbf{\Upsilon}(.)$ is not a diffeormorphism. But, then, the best responses do not exist. Therefore, the reduced form payoffs of leaders do not exist. Then, there is no strategic equilibrium which is the solution to the two-stage game. Nevertheless, there is a CNE which corresponds to the autarkic CE.

4.4. SNE without differentiability

Let $T_X = \{1, 2\}$ and $T_Y = \{1, 2\}$, with:

$$u_k(x_k, y_k) = \min\{x_k, y_k\}, \ k = i, j, \ i, j = 1, \tag{56}$$

$$u_k(x_k, y_k) = x_k + y_k, \ k = i, j, \ i, j = 2. \tag{57}$$

The CE supplies are given by $(q_1^*, q_2^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $(b_1^*, b_2^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$. The CNE supplies are $(\hat{q}_1, \hat{q}_2) = (0, 0)$ and $(\hat{b}_1, \hat{b}_2) = (0, 0)$.

Let us now compute the SNE. The OM are given by:

$$\phi_2(q_1; b_1, b_2) = -q_1 + \sqrt{(b_1 + b_2)q_1}, \tag{58}$$

$$\psi_2(q_1, q_2; b_1) = -b_1 + \sqrt{b_1(q_1 + q_2)}. \tag{59}$$

Let $\mathbf{\Upsilon}(\Phi_2(.); \Psi_2(.)) = \mathbf{0}$, where $\Phi_2(q_1, q_2; b_1, b_2) := q_1 + q_2 - \sqrt{(b_1 + b_2)q_1}$ and $\Psi_2(q_1, q_2; b_1, b_2) := b_1 + b_2 - \sqrt{b_1(q_1 + q_2)}$. The Jacobian is given by:

$$\mathcal{J}_{\mathbf{\Upsilon}_{(\mathbf{q}^F;\mathbf{b}^F)}} = \begin{bmatrix} 1 & -\frac{1}{2}\sqrt{\frac{q_1}{b_1+b_2}} \\ -\frac{1}{2}\sqrt{\frac{b_1}{q_1+q_2}} & 1 \end{bmatrix}. \tag{60}$$

We get $\left|\mathcal{J}_{\mathbf{\Upsilon}_{(\mathbf{q}^F;\mathbf{b}^F)}}\right| = 1 - \frac{1}{4}\sqrt{\frac{b_1 q_1}{(b_1+b_2)(q_1+q_2)}} \neq 0$, so there exist best responses:

$$\sigma_2(q_1; b_1) = \frac{\sqrt{(4b_1 - 3q_1)q_1} - q_1}{2}, \tag{61}$$

$$\varphi_2(q_1; b_1) = \frac{\sqrt{(4q_1 - 3b_1)b_1} - b_1}{2}. \tag{62}$$

In the first stage, any leader maximizes her reduced form payoff, with $p_X(q_1; b_1) = \frac{b_1+\varphi_2(q_1;b_1)}{q_1+\sigma_2(q_1;b_1)} = \frac{\sqrt{(4q_1-3b_1)b_1}+b_1}{\sqrt{(4b_1-3q_1)q_1}+q_1}$, so the problems of the two leaders may be written:

$$\max_{\{q_1\}} \ \min \left(1 - q_1, \frac{\sqrt{(4q_1 - 3b_1)b_1} + b_1}{\sqrt{(4b_1 - 3q_1)q_1} + q_1} q_1 \right), \tag{63}$$

$$\max_{\{b_1\}} \ \min \left(\frac{\sqrt{(4b_1 - 3q_1)q_1} + q_1}{\sqrt{(4q_1 - 3b_1)b_1} + b_1} b_1, 1 - b_1 \right). \tag{64}$$

Then, some computations lead to the unique SNE strategy profile:

$$(\tilde{q}_1, \tilde{q}_2) = \left(\frac{1}{2}, 0\right) \tag{65}$$

$$(\tilde{b}_1, \tilde{b}_2) = \left(\frac{1}{2}, 0\right). \tag{66}$$

A SNE with trade may exist even if Assumptions (2a), (2c) and (2d) are not satisfied for all traders: Assumption 2 constitutes a set of sufficient conditions. But, beyond this, Example 4 provides new insights on competition in bilateral oligopolies. The main salient feature stems from the fact that the symmetric CNE is autarkic, whilst the SNE is non-autarkic. Indeed, this example allows trade in the subgame *between leaders* whilst there is no trade in the subgame between followers, and thereby in the entire game betweeen leaders and followers. It should be noted that only the followers would have made trade if the specified utility functions had been reversed, that is if $u_k(x_k, y_k) = x_k + y_k$, $k = i, j$, $i, j = 1$, and $u_k(x_k, y_k) = \min\{x_k, y_k\}$, $k = i, j$, $i, j = 2$, then the SNE supplies are given by $(\tilde{q}_1, \tilde{q}_2) = (0, \frac{1}{2})$ and $(\tilde{b}_1, \tilde{b}_2) = (0, \frac{1}{2})$. Such cases, which are specific to a sequential strategic market game, might be called either a "partial trade equilibrium" or a "partial autarkic equilibrium".

5. CONCLUSION

We considered a framework in which all traders, consumers and suppliers, behaved strategically. Our model provided a rich set of strategic interactions, and thereby it offered new insights on the study of optimal behavior in oligopolistic decentralized markets. As a sequential game, it illustrated the possibility that trade can only take place in one subgame. It also showed that the existence of a Nash equilibrium for the entire game also depended on whether the followers' best responses were consistent. Assumptions 1 and 2 were sufficient conditions to ensure that the system of equations which determined the followers' strategies were a C^2-diffeomorphism, and thereby to show the existence of a SNE with trade.

Further theoretical issues could be explored. First, the existence of a SNE should be extended to the case with more than two stages, and/or to an exchange economy with a number of commodities larger than two. Second, the endogeneization of the order of moves should be undertaken.

6. APPENDIX

Through this Appendix, we prove some intermediary results which are useful to show our Theorem. Appendix A is devoted to the study of the followers' best responses. Appendix B deals with the monotonicity properties of such mappings. In Appendix C, we show that the existence of unique strategies. In Appendix D, we show that the followers' best responses are bounded. Appendix E deals with the existence of leaders' best responses. Appendix F shows the market price is bounded in an ϵ-SNE.

6.1. Appendix A: Proof of Proposition 1

Consider a follower of type X. We show the best reply $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$ *exists* and is a $\mathcal{C}^2$ *function.* To this end, we study the properties of the payoff function $\pi_{i,\epsilon}(.)$. Then, we characterize the solution with the Kuhn-Tucker conditions.

Given an admissible strategy profile for all leaders and for all admissible strategy profiles of other followers, the problem of follower i consists of maximizing his payoff $\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon)$ subject to the set of admissible strategies $\mathcal{S}_i$. The solution, if it exists, is the follower i's best reply given in Definition 1, namely $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$. Given $(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F) \in \mathbf{S}_{-i}$, the problem for follower i may be written:

$$\max_{\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)} \{\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F) : q_{i,\epsilon} \in \mathcal{S}_i\},\ \epsilon > 0, \tag{A1}$$

where $\mathcal{S}_i = [0, \alpha_i]$ is a nonempty compact convex set, and $\pi_{i,\epsilon}(.)$ is a continuous function of $(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$ as $\pi_{i,\epsilon}(.) \in \mathcal{C}^2$. We show that $\pi_{i,\epsilon}(.)$ is a strictly quasi-concave function of $q_{i,\epsilon}$. Differentiating (6) with respect to $q_{i,\epsilon}$ leads to:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} = -\frac{\partial u_i}{\partial x_i} + p_X^\epsilon \frac{Q_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + Q_{-i,\epsilon} + \epsilon} \frac{\partial u_i}{\partial y_i}. \tag{A2}$$

Differentiating (A2) with respect to $q_{i,\epsilon}$ leads to:

$$\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} = -\left|\bar{\mathcal{H}}_{u_{\mathbf{z}_i}}\right| - 2p_X^\epsilon \frac{Q_{-i,\epsilon} + \epsilon}{(Q_\epsilon + \epsilon)^2} \frac{\partial u_i}{\partial y_i}, \tag{A3}$$

where $\left|\bar{\mathcal{H}}_{u_{\mathbf{z}_i}}\right| = \frac{\partial^2 u_i}{(\partial x_i)^2} - 2p_X^\epsilon \frac{Q_{-i,\epsilon}+\epsilon}{Q_\epsilon+\epsilon}\frac{\partial^2 u_i}{\partial x_i \partial y_i} + \left(p_X^\epsilon \frac{Q_{-i,\epsilon}+\epsilon}{Q_\epsilon+\epsilon}\right)^2 \frac{\partial^2 u_i}{(\partial y_i)^2}$ is the determinant of the bordered Hessian matrix of the function u_i (see Assumption 2c). As $p_X^\epsilon \frac{Q_{-i,\epsilon}+\epsilon}{Q_\epsilon+\epsilon} = \frac{\partial u_i/\partial x_i}{\partial u_i/\partial y_i}$, and $\left|\bar{\mathcal{H}}_{u_{\mathbf{z}_i}}\right| > 0$ (Assumption 2c), and the last term is negative, then $\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} < 0$, so $\pi_i^\epsilon(.)$ is strictly concave (thereby strictly quasi-concave) of $q_{i,\epsilon}$. But then, the solution to (A1) is unique, so $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$ is *point-valued*. Then, for each $i = m_X+1, ..., n_X$, the mapping $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$ is a function.

As the objective $\pi_{i,\epsilon}(.)$ is strictly quasi-concave (condition (d) in Arrow and Enthoven (1961) holds), and the constraint set is quasi-convex (as it is convex), the Kuhn-Tucker conditions are sufficient to identify the solution to (A1). Define the Lagrangian $\mathcal{L}_i^\epsilon : \mathbf{S} \times \mathbb{R}_+^2 \times \mathbb{R}_{++} \to \mathbb{R}$, with $(q_{i,\epsilon}, \mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \lambda_{i,\epsilon}, \mu_{i,\epsilon}; \epsilon) \mapsto \mathcal{L}_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \lambda_{i,\epsilon}, \mu_{i,\epsilon}; \epsilon)$, as:

$$\mathcal{L}_i^\epsilon(.;\epsilon) := \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) + \lambda_{i,\epsilon}(\alpha_i - q_{i,\epsilon}) + \mu_{i,\epsilon} q_{i,\epsilon},\ \epsilon > 0, \tag{A4}$$

where $\lambda_{i,\epsilon} \geqslant 0$ and $\mu_{i,\epsilon} \geqslant 0$ are the Kuhn-Tucker multipliers. Then, for all $\varepsilon > 0$, and given $(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F) \in \mathbf{S}_{-i}$, $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$, is the unique solution to:

$$\max_{\phi_i^\epsilon(.)} \mathcal{L}_i^\epsilon(.;\epsilon) = u_i(\alpha_i - q_{i,\epsilon}, \frac{B_\epsilon + \epsilon}{q_{i,\epsilon} + Q_{-i,\epsilon} + \epsilon} q_{i,\epsilon}) + \lambda_{i,\epsilon}(\alpha_i - q_{i,\epsilon}) + \mu_{i,\epsilon} q_{i,\epsilon}. \tag{A5}$$

For all $\epsilon > 0$, by using (A2), the Kuhn-Tucker conditions may be written:

$$\frac{\partial \mathcal{L}_i^\epsilon}{\partial q_i} = -\frac{\partial u_i}{\partial x_i} + p_X^\epsilon \frac{Q_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + Q_{-i,\epsilon} + \epsilon} \frac{\partial u_i}{\partial y_i} - \lambda_{i,\epsilon} + \mu_{i,\epsilon} = 0, \tag{A6}$$

$$\lambda_{i,\epsilon} \geqslant 0,\ (\alpha_i - q_{i,\epsilon}) \geqslant 0, \text{ with } \lambda_{i,\epsilon}(\alpha_i - q_{i,\epsilon}) = 0, \tag{A7}$$

$$\mu_{i,\epsilon} \geqslant 0,\ q_{i,\epsilon} \geqslant 0, \text{ with } \mu_{i,\epsilon} q_{i,\epsilon} = 0. \tag{A8}$$

Therefore, if $q_{i,\varepsilon} > 0$, then $\mu_{i,\epsilon} = 0$, where $b_{i,\varepsilon}$ is the solution to:

$$-\frac{\partial u_i}{\partial x_i} + p_X^\epsilon \frac{Q_{-i,\epsilon} + \epsilon}{q_{i,\epsilon} + Q_{-i,\epsilon} + \epsilon} \frac{\partial u_i}{\partial y_i} = \lambda_{i,\epsilon}, \tag{A9}$$

which yields $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) > 0$. In addition, if $\lambda_{i,\epsilon} > 0$, then we have $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) = \alpha_i$, while if $\lambda_{i,\epsilon} = 0$, then we have $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) \in (0, \alpha_i)$. Now, if $\mu_{i,\epsilon} > 0$, then $q_{i,\varepsilon} = 0$, which means that $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) = 0$ and $\lambda_{i,\epsilon} = 0$ since $q_{i,\varepsilon} < \alpha_i$. Therefore, either we have $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) > 0$ when $q_{i,\epsilon} \in (0, \alpha_i]$ or $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) = 0$. Then, $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon) \geqslant 0$. In each case there exists a unique solution to (A1): either $\phi_i^\epsilon(.) \in \{0, \alpha_i\}$ or $\phi_i^\epsilon(.) \in (0, \alpha_i)$, $i = m_X+1, ..., n_X$.

Finally, we show that $\phi_i^\epsilon(.) \in \mathcal{C}^2$. (A6) defines implicitly $\phi_i^\epsilon(.)$. As $\pi_i^\epsilon(.) \in \mathcal{C}^2$ and $\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} \neq 0$, from the Implicit Function Theorem, we deduce $\phi_i^\epsilon(.) \in \mathcal{C}^2$.

6.2. Appendix B: Proof of Proposition 2

Let $\bar{\mathbf{a}} = (\bar{\mathbf{q}}_\epsilon^L, \bar{\mathbf{q}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L); \bar{\mathbf{b}}_\epsilon^L, \bar{\mathbf{b}}_\epsilon^F(\bar{\mathbf{q}}_\epsilon^L; \bar{\mathbf{b}}_\epsilon^L)) \in \mathbf{S}$. First: $\mathcal{J}_{\phi^\epsilon_{\mathbf{q}_\epsilon^F}}(\bar{\mathbf{a}}) \in (-\mathbf{I}, \mathbf{I})$, where $\mathbf{I}$ is the $(n_X - m_X, n_X - m_X)$ unit matrix.

The matrix $\mathcal{J}_{\phi^\epsilon_{\mathbf{q}_\epsilon^F}}(\bar{\mathbf{a}})$ has unit terms on its main diagonal. To study the partial effects of a change in the strategy of any other follower, i.e. $q_{-i,\epsilon}$, for $-i \neq i$, $-i = m_X + 1, ..., n_X$, and $b_{j,\epsilon}$, $j = m_Y + 1, ..., n_Y$, consider the identity:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}}(\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon), \boldsymbol{\phi}_{-i}^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{i,\epsilon}^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) \equiv 0, \tag{B1}$$

where, for each $i = m_X + 1, ..., n_X$, and $\phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F; \epsilon)$ is the solution to (A2). Implicit partial differentiation of (B1) with respect to $q_{-i,\epsilon}$, for $-i \neq i$, leads to $\frac{\partial \phi_i^\epsilon(.)}{\partial q_{-i,\epsilon}} = -\frac{\frac{\partial^2 \pi_i^\epsilon}{\partial q_{i,\epsilon} \partial q_{-i,\epsilon}}}{\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2}}$, so we deduce:

$$\frac{\partial \phi_i^\epsilon(.)}{\partial q_{-i,\epsilon}} = \frac{p_X^\epsilon\left(\frac{q_{i,\epsilon}}{Q_\epsilon + \epsilon}\frac{\partial^2 u_i}{\partial x_i \partial y_i} + \frac{q_{i,\epsilon} - (Q_{-i,\epsilon} + \epsilon)}{(Q_\epsilon + \epsilon)^2}\frac{\partial u_i}{\partial y_i} - p_X^\epsilon \frac{(Q_{-i,\epsilon} + \epsilon) q_{i,\epsilon}}{(Q_\epsilon + \epsilon)^2}\frac{\partial^2 u_i}{(\partial y_i)^2}\right)}{\frac{\partial^2 u_i}{(\partial x_i)^2} - p_X^\epsilon\left(2\frac{Q_{-i,\epsilon} + \epsilon}{Q_\epsilon + \epsilon}\frac{\partial^2 u_i}{\partial x_i \partial y_i} - p_X^\epsilon\left(\frac{Q_{-i,\epsilon} + \epsilon}{Q_\epsilon + \epsilon}\right)^2 \frac{\partial^2 u_i}{(\partial y_i)^2} + 2\frac{Q_{-i,\epsilon} + \epsilon}{(Q_\epsilon + \epsilon)^2}\frac{\partial u_i}{\partial y_i}\right)}. \tag{B2}$$

Assume, without loss of generality, that, for at least one leader i or one follower i' we have $\tilde{q}_{i,\epsilon} \leqslant \frac{\tilde{Q}_\epsilon}{2}$ or $\tilde{q}_{i',\epsilon} \leqslant \frac{\tilde{Q}_\epsilon}{2}$ (otherwise $\tilde{q}_{i,\epsilon} + \tilde{q}_{i',\epsilon} > \tilde{Q}_\epsilon$). As $\left|\frac{(Q_{-i,\epsilon} + \epsilon) q_{i,\epsilon}}{(Q_\epsilon + \epsilon)^2}\right| < \left|\left(\frac{Q_{-i,\epsilon} + \epsilon}{Q_\epsilon + \epsilon}\right)^2\right|$, $2\frac{Q_{-i,\epsilon} + \epsilon}{Q_\epsilon + \epsilon} > \frac{q_{i,\epsilon}}{Q_\epsilon + \epsilon}$, and $\frac{q_{i,\epsilon} - (Q_{-i,\epsilon} + \epsilon)}{(Q_\epsilon + \epsilon)^2} < 2\frac{Q_{-i,\epsilon} + \epsilon}{(Q_\epsilon + \epsilon)^2}$, then $\left|\frac{\partial \phi_i^\epsilon(.)}{\partial q_{-i,\epsilon}}\right| < 1$.

Second: $\mathcal{J}_{\phi^\epsilon_{\mathbf{b}_\epsilon^F}}(\bar{\mathbf{a}}) \in (-\mathbf{I}, \mathbf{I})$. Implicit partial differentiation of (B1) with respect to $b_{j,\epsilon}$, $j = m_Y + 1, ..., n_Y$, leads to:

$$\frac{\partial \phi_i^\epsilon(.)}{\partial b_{j,\epsilon}} = \frac{\frac{q_{i,\epsilon}}{Q_\epsilon + \epsilon}\frac{\partial^2 u_i}{(\partial x_i)^2} - \frac{Q_{-i,\epsilon} + \epsilon}{(\mathbf{Q}_\epsilon + \epsilon)^2}\frac{\partial u_i}{\partial y_i} - p_X^\epsilon \frac{(Q_{-i,\epsilon} + \epsilon) q_{i,\epsilon}}{(\mathbf{Q}_\epsilon + \epsilon)^2}\frac{\partial^2 u_i}{(\partial y_i)^2}}{\frac{\partial^2 u_i}{(\partial x_i)^2} - p_X^\epsilon\left(2\frac{Q_{-i,\epsilon} + \epsilon}{Q_\epsilon + \epsilon}\frac{\partial^2 u_i}{\partial x_i \partial y_i} - p_X^\epsilon\left(\frac{Q_{-i,\epsilon} + \epsilon}{Q_\epsilon + \epsilon}\right)^2 \frac{\partial^2 u_i}{(\partial y_i)^2} + 2\frac{Q_{-i,\epsilon} + \epsilon}{(Q_\epsilon + \epsilon)^2}\frac{\partial u_i}{\partial y_i}\right)}. \tag{B3}$$

Then, a similar reasoning leads to the conclusion $\left|\frac{\partial \phi_i^\epsilon(.)}{\partial b_{j,\epsilon}}\right| < 1$, for all $i \in \{m_X + 1, ..., n_X\}$, and all $j \in \{m_Y + 1, ..., n_Y\}$.

6.3. Appendix C: Proof of Lemma 1

Consider the set of best responses specified in Definition 1. To build the system of equations that implicitly define the followers' strategies for the perturbed game, define the function $\Phi_i^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathcal{S}_i$, with $\Phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) := q_{i,\epsilon}(.) - \phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_{-i,\epsilon}^F(.), ; \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon)$, $i = m_X + 1, ..., n_X$, and the function $\Psi_j^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathcal{S}_j$, with $\Psi_j^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) := b_{j,\epsilon}(.) - \psi_j^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_{-j,\epsilon}^F(.); \epsilon)$, $j = m_Y + 1, ..., n_Y$. As for each $i = m_X + 1, ..., n_X$, $\phi_i^\epsilon(.) \in \mathcal{C}^2(.)$, then $\Phi_i^\epsilon(.) \in \mathcal{C}^2$, $i = m_X + 1, ..., n_X$. Likewise, $\Psi_j^\epsilon(.) \in \mathcal{C}^2$, $j = m_Y + 1, ..., n_Y$. For all $\epsilon > 0$, consider the system of equations for $\mathbf{\Gamma}^\epsilon$:

$$\begin{aligned} \Phi_i^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) &= 0, \, i = m_X + 1, ..., n_X, \\ \Psi_j^\epsilon(\mathbf{q}_\epsilon^L, \mathbf{q}_\epsilon^F(.); \mathbf{b}_\epsilon^L, \mathbf{b}_\epsilon^F(.); \epsilon) &= 0, \, j = m_Y + 1, ..., n_Y. \end{aligned} \tag{C1}$$

Define the $(n_X - m_X + n_Y - m_Y)$-dimensional vector function $\boldsymbol{\Upsilon}^\epsilon : \mathbf{S} \times \mathbb{R}_{++} \to \mathbf{S}^F$, $\boldsymbol{\Upsilon}^\epsilon(\mathbf{q}^L_\epsilon, \mathbf{q}^F_\epsilon(.); \mathbf{b}^L_\epsilon, \mathbf{b}^F_\epsilon(.); \epsilon) = (\Phi^\epsilon_{m_X+1}(.;\epsilon), ..., \Phi^\epsilon_{n_X}(.;\epsilon); \Psi^\epsilon_{m_Y+1}(.;\epsilon), ..., \Psi^\epsilon_{n_Y}(.;\epsilon))$. Then, (C1) may be written as a $(n_X - m_X + n_Y - m_Y)$-dimensional vector equation:

$$\boldsymbol{\Upsilon}^\epsilon(\mathbf{q}^L_\epsilon, \mathbf{q}^F_\epsilon(.); \mathbf{b}^L_\epsilon, \mathbf{b}^F_\epsilon(.); \epsilon) = \mathbf{0}. \tag{C2}$$

Since we focus on inner solutions, consider the restriction of $\mathbf{S} \times \mathbb{R}_{++}$ to the open set $\bar{\mathbf{S}} \times \mathbb{R}_{++}$, with $\bar{\mathcal{S}}_i \subset \mathcal{S}_i$, $i \in T_X$, and $\bar{\mathcal{S}}_j \subset \mathcal{S}_j$, $j \in T_Y$. The vector function $\boldsymbol{\Upsilon}^\epsilon(\mathbf{q}^L_\epsilon, \mathbf{q}^F_\epsilon(.); \mathbf{b}^L_\epsilon, \mathbf{b}^F_\epsilon(.); \epsilon)$ is $\mathcal{C}^2$ on the open set $\bar{\mathbf{S}} \times \mathbb{R}_{++}$ as each Φ^ϵ_i and each Ψ^ϵ_j are $\mathcal{C}^2$ functions of $(\mathbf{q}^L_\epsilon; \mathbf{b}^L_\epsilon)$ on the open set $\bar{\mathbf{S}} \times \mathbb{R}_{++}$. Let $\bar{\mathbf{a}} = (\bar{\mathbf{q}}^L_\epsilon, \bar{\mathbf{q}}^F_\epsilon(\bar{\mathbf{q}}^L_\epsilon; \bar{\mathbf{b}}^L_\epsilon); \bar{\mathbf{b}}^L_\epsilon, \bar{\mathbf{b}}^F_\epsilon(\bar{\mathbf{q}}^L_\epsilon; \bar{\mathbf{b}}^L_\epsilon))$ be an interior point of $\mathbf{S}$, where $(\bar{\mathbf{q}}^L_\epsilon; \bar{\mathbf{b}}^L_\epsilon)$ corresponds to a parameter configuration. Therefore, the following identity, which defines implicitly (at least locally) the strategies $\mathbf{q}^L_\epsilon := \boldsymbol{\sigma}^\epsilon(.)$ and $\mathbf{b}^L_\epsilon := \boldsymbol{\varphi}^\epsilon(.)$, holds in an open neighborhood of $\bar{\mathbf{a}}$:

$$\boldsymbol{\Upsilon}^\epsilon(\mathbf{q}^L_\epsilon, \boldsymbol{\sigma}^\epsilon(\mathbf{b}^L_\epsilon; \mathbf{q}^L_\epsilon; \epsilon); \mathbf{b}^L_\epsilon, \boldsymbol{\varphi}^\epsilon(\mathbf{b}^L_\epsilon; \mathbf{q}^L_\epsilon; \epsilon)) \equiv \mathbf{0}. \tag{C3}$$

We now show that $\boldsymbol{\Upsilon}^\epsilon$ is a local $\mathcal{C}^2$-diffeomorphism, i.e. there exists a product of open sets $\mathcal{U} \times \mathcal{V}$ in $\bar{\mathbf{S}}$ and a product neighborhood $(\mathcal{U}_L \times \mathcal{V}_L)$ in $\prod_{i=1}^{m_X} \bar{\mathcal{S}}_i \times \prod_{j=1}^{m_Y} \bar{\mathcal{S}}_j$, with $\bar{\mathbf{a}} \subseteq \mathcal{U} \times \mathcal{V}$ and $(\mathbf{q}^L_\epsilon; \mathbf{b}^L_\epsilon) \subseteq (\mathcal{U}_L \times \mathcal{V}_L)$ such that for each $(\mathbf{q}^L_\epsilon; \mathbf{b}^L_\epsilon)$ in $(\mathcal{U}_L \times \mathcal{V}_L)$, there exists (at least locally) one *unique* $(n_X - m_X + n_Y - m_Y)$ dimensional $\mathcal{C}^2$ vector function $(\boldsymbol{\sigma}^\epsilon(\mathbf{q}^L_\epsilon; \mathbf{b}^L_\epsilon; \epsilon); \boldsymbol{\varphi}^\epsilon(\mathbf{q}^L_\epsilon; \mathbf{b}^L_\epsilon; \epsilon)) = [\boldsymbol{\Upsilon}^\epsilon]^{-1}(\mathbf{0})$ in some neighborhood of $(\mathbf{q}^L_\epsilon; \mathbf{b}^L_\epsilon)$ such that $(\mathbf{q}^L_\epsilon, \boldsymbol{\sigma}^\epsilon(.); \mathbf{b}^L_\epsilon, \boldsymbol{\varphi}^\epsilon(.)) \in \mathcal{U} \times \mathcal{V}$ and (C3) holds.

Implicit partial differentiation with respect to each component of $(\bar{\mathbf{q}}^L_\epsilon; \bar{\mathbf{b}}^L_\epsilon)$ yields:

$$\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon; \mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}}).\mathcal{A}^\epsilon + \mathcal{B}^\epsilon = \mathbf{0}, \text{ for each } \epsilon > 0, \tag{C4}$$

where:

$$\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon; \mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}}) = \begin{bmatrix} 1 & \cdots & \frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{n_X,\epsilon}} & \frac{\partial \Phi^\epsilon_{m_X+1}}{\partial b_{m_Y+1,\epsilon}} & \cdots & \frac{\partial \Phi^\epsilon_{m_X+1}}{\partial b_{n_Y,\epsilon}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Phi^\epsilon_{n_X}}{\partial q_{m_X+1,\epsilon}} & \cdots & 1 & \frac{\partial \Phi^\epsilon_{n_X}}{\partial b_{m_Y+1,\epsilon}} & \cdots & \frac{\partial \Phi^\epsilon_{n_X}}{\partial b_{n_Y,\epsilon}} \\ \frac{\partial \Psi^\epsilon_{m_Y+1}}{\partial q_{m_X+1,\epsilon}} & \cdots & \frac{\partial \Psi^\epsilon_{m_Y+1}}{\partial q_{n_X,\epsilon}} & 1 & \cdots & \frac{\partial \Psi^\epsilon_{m_Y+1}}{\partial b_{n_Y,\epsilon}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Psi^\epsilon_{n_Y}}{\partial q_{m_X+1,\epsilon}} & \cdots & \frac{\partial \Psi^\epsilon_{n_Y}}{\partial q_{n_X,\epsilon}} & \frac{\partial \Psi^\epsilon_{n_Y}}{\partial b_{m_Y+1,\epsilon}} & \cdots & 1 \end{bmatrix}$$

is an $(n_X - m_X + n_Y - m_Y, n_X - m_X + n_Y - m_Y)$ matrix, while

$$\mathcal{A}^\epsilon = \begin{bmatrix} \frac{\partial q_{m_X+1,\epsilon}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial q_{m_X+1,\epsilon}}{\partial q_{m_X,\epsilon}} & \frac{\partial q_{m_X+1,\epsilon}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial q_{m_X+1,\epsilon}}{\partial b_{m_Y,\epsilon}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial q_{n_X,\epsilon}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial q_{n_X,\epsilon}}{\partial q_{m_X,\epsilon}} & \frac{\partial q_{n_X,\epsilon}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial q_{n_X,\epsilon}}{\partial b_{m_Y,\epsilon}} \\ \frac{\partial b_{m_X+1,\epsilon}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial b_{m_X+1,\epsilon}}{\partial q_{m_X,\epsilon}} & \frac{\partial b_{m_Y+1,\epsilon}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial b_{m_Y+1,\epsilon}}{\partial b_{m_Y,\epsilon}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial b_{n_Y,\epsilon}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial b_{n_Y,\epsilon}}{\partial q_{m_X,\epsilon}} & \frac{\partial b_{n_Y,\epsilon}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial b_{n_Y,\epsilon}}{\partial b_{m_Y,\epsilon}} \end{bmatrix}$$

and

$$\mathcal{B}^{\epsilon}=\begin{bmatrix} \frac{\partial \Phi^{\epsilon}_{m_X+1,\epsilon}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Phi^{\epsilon}_{m_X+1,\epsilon}}{\partial q_{m_X,\epsilon}} & \frac{\partial \Phi^{\epsilon}_{m_X+1,\epsilon}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial \Phi^{\epsilon}_{m_X+1,\epsilon}}{\partial b_{m_Y,\epsilon}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Phi^{\epsilon}_{n_X,\epsilon}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Phi^{\epsilon}_{n_X,\epsilon}}{\partial q_{m_X,\epsilon}} & \frac{\partial \Phi^{\epsilon}_{n_X,\epsilon}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial \Phi^{\epsilon}_{n_X,\epsilon}}{\partial b_{m_Y,\epsilon}} \\ \frac{\partial \Psi^{\epsilon}_{m_Y+1}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Psi^{\epsilon}_{m_Y+1}}{\partial q_{m_X,\epsilon}} & \frac{\partial \Psi^{\epsilon}_{m_Y+1}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial \Psi^{\epsilon}_{m_Y+1}}{\partial b_{m_Y,\epsilon}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Psi^{\epsilon}_{n_Y}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Psi^{\epsilon}_{n_Y}}{\partial q_{m_X,\epsilon}} & \frac{\partial \Psi^{\epsilon}_{n_Y}}{\partial b_{1,\epsilon}} & \cdots & \frac{\partial \Psi^{\epsilon}_{n_Y}}{\partial b_{m_Y,\epsilon}} \end{bmatrix}$$

are matrices of dimension $(n_X - m_X + n_Y - m_Y, m_X + m_Y)$.

The square matrix $\mathcal{J}_{\boldsymbol{\Upsilon}^{\epsilon}_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})$ has unit terms on the main diagonal and each off-diagonal term is bounded below by -1 and above by 1, as from Proposition 2, we have that $-\mathbf{I} << \mathcal{J}_{\boldsymbol{\phi}^{\epsilon}_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}} << \mathbf{I}$ and $-\mathbf{I} << \mathcal{J}_{\boldsymbol{\psi}^{\epsilon}_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}} << \mathbf{I}$. Then, $\frac{\partial \Phi^{\epsilon}_i(.)}{\partial q_{-i,\epsilon}} = -\frac{\partial \phi^{\epsilon}_i(.)}{\partial q_{-i,\epsilon}} \in (-1,1)$, with $-i \neq i$, and $\left|\frac{\partial \Phi^{\epsilon}_i(.)}{\partial b_{j,\epsilon}}\right| = \left|-\frac{\partial \phi^{\epsilon}_i(.)}{\partial b_{j,\epsilon}}\right| < 1$, $i = m_X+1,...,n_X$; and $\frac{\partial \Psi^{\epsilon}_j(.)}{\partial b_{-j,\epsilon}} = -\frac{\partial \psi^{\epsilon}_j(.)}{\partial b_{-j,\epsilon}} \in (-1,1)$, with $-j \neq j$, and $\left|\frac{\partial \Psi^{\epsilon}_j(.)}{\partial q_{i,\epsilon}}\right| = \left|-\frac{\partial \psi^{\epsilon}_j(.)}{\partial q_{i,\epsilon}}\right| < 1$, $j = m_Y+1,...,n_Y$. The signs of the off diagonal terms depend on whether the strategies of followers are complements or substitutes. But, in any case, for all $\epsilon > 0$, the rows of the matrix $\mathcal{J}_{\boldsymbol{\Upsilon}^{\epsilon}_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{q}}_\varepsilon;\bar{\mathbf{b}}_\varepsilon)$ are linearly independent, so the matrix $\mathcal{J}_{\boldsymbol{\Upsilon}^{\epsilon}_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})$ is of full rank, and then invertible. Then, for all $\epsilon > 0$, $\left|\mathcal{J}_{\boldsymbol{\Upsilon}^{\epsilon}_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right| \neq 0$: the differential of $\boldsymbol{\Upsilon}^{\epsilon}$ is a $\mathcal{C}^1$-diffeomorphism, and, by the local inversion theorem, $\boldsymbol{\Upsilon}^{\epsilon}$ is a local $\mathcal{C}^2$-diffeomorphism. But, then, by the Implicit Function Theorem (Raeburn 1979, Dontchev and Rockafellar 2014), there exist open sets $\mathcal{U} \times \mathcal{V}$ in $\bar{\mathbf{S}}$ and $(\mathcal{U}_L \times \mathcal{V}_L)$ in $\prod_{i=1}^{m_X} \bar{\mathcal{S}}_i \times \prod_{j=1}^{m_Y} \bar{\mathcal{S}}_j$, with $\bar{\mathbf{a}} \subseteq \mathcal{U} \times \mathcal{V}$ and $(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon) \subseteq (\mathcal{U}_L \times \mathcal{V}_L)$ such that for each $(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon)$ in $(\mathcal{U}_L \times \mathcal{V}_L)$, there exists (at least locally) one *unique* $(n_X - m_X + n_Y - m_Y)$ dimensional vector function $(\mathbf{q}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon);\mathbf{b}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon))$ in some neighborhood of $(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon)$ such that $(\mathbf{q}^L_\epsilon, \mathbf{q}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon);\mathbf{b}^L_\epsilon, \mathbf{b}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)) \in \mathcal{U} \times \mathcal{V}$, and the identity $\boldsymbol{\Upsilon}^{\epsilon}(\mathbf{q}^L_\epsilon, \mathbf{q}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon);\mathbf{b}^L_\epsilon, \mathbf{b}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)) \equiv \mathbf{0}$ holds. Indeed, the unique solution $(\boldsymbol{\sigma}^{\epsilon}(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon);\boldsymbol{\varphi}^{\epsilon}(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon))$ to $(\mathbf{q}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon);\mathbf{b}^F_\epsilon(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)) = [\boldsymbol{\Upsilon}^{\epsilon}]^{-1}(\mathbf{0})$ is defined by $\boldsymbol{\sigma}^{\epsilon} : \mathcal{S}^L \times \mathbb{R}_{++} \supset (\mathcal{U}_L \times \mathcal{V}_L) \to \prod_{i=m_X+1}^{n_X} \mathcal{S}_i$, with $\mathbf{q}^F_\epsilon = \boldsymbol{\sigma}^{\epsilon}(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)$, and by $\boldsymbol{\varphi}^{\epsilon} : \mathcal{S}^L \times \mathbb{R}_{++} \supset (\mathcal{U}_L \times \mathcal{V}_L) \to \prod_{j=m_Y+1}^{n_Y} \mathcal{S}_j$, with $\mathbf{b}^F_\epsilon = \boldsymbol{\varphi}^{\epsilon}(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)$. For all $\epsilon > 0$, each component function $\sigma^{\epsilon}_i(.)$ is defined as $\sigma^{\epsilon}_i : \mathcal{S}^L \times \mathbb{R}_{++} \supset (\mathcal{U}_L \times \mathcal{V}_L) \to \mathcal{S}_i$, with $q_{i,\epsilon} = \sigma^{\epsilon}_i(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)$, $i = m_X+1,...,n_X$. The same holds for $\varphi^{\epsilon}_j : \mathcal{S}^L \times \mathbb{R}_{++} \supset (\mathcal{U}_L \times \mathcal{V}_L)$, with $b_{j,\epsilon} = \varphi^{\epsilon}_j(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)$ $j = m_Y+1,...,n_Y$. In addition, for all $\epsilon > 0$, $\sigma^{\epsilon}_i(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon) \in \mathcal{C}^2(\mathbf{S}, \mathcal{S}_i)$, for each $i = m_X+1,...,n_X$, and $\varphi^{\epsilon}_j(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon) \in \mathcal{C}^2(\mathbf{S}, \mathcal{S}_j)$, for each $j = m_Y+1,...,n_Y$.

6.4. Appendix D: Proof of Proposition 3

Let $\bar{\mathbf{a}} = (\bar{\mathbf{q}}^L_\epsilon, \bar{\mathbf{q}}^F_\epsilon(\bar{\mathbf{q}}^L_\epsilon;\bar{\mathbf{b}}^L_\epsilon);\bar{\mathbf{b}}^L_\epsilon, \bar{\mathbf{b}}^F_\epsilon(\bar{\mathbf{q}}^L_\epsilon;\bar{\mathbf{b}}^L_\epsilon))$ in $\mathbf{S}$. We show that, for each $i = m_X+1,...,n_X$, $\sigma^{\epsilon}_i(.)$ satisfies $-1 \leqslant \frac{\partial \sigma^{\epsilon}_i(\mathbf{q}^L_\epsilon;\mathbf{b}^L_\epsilon;\epsilon)}{\partial q_{i,\epsilon}} < 1$, and, for each $j = m_Y+1,...,n_Y$, $\varphi^{\epsilon}_j(.)$ satisfies $\frac{\partial \varphi^{\epsilon}_j(.)}{\partial q_{i,\epsilon}} \geqslant 0$.

First, we show $-1 \leqslant \frac{\partial \sigma_i^\epsilon(.)}{\partial q_{i,\epsilon}} < 1$. Consider (9). From Cramer's rule:

$$\frac{\partial q_{m_X+1,\epsilon}}{\partial q_{1,\epsilon}} = -\frac{\left|\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right|}{\left|\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right|}, \tag{D1}$$

where $\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{b}^F_\epsilon;\mathbf{q}^F_\epsilon)}}(\bar{\mathbf{a}})$ is the $(n_X - m_X + n_Y - m_Y, n_X - m_X + n_Y - m_Y)$ square matrix obtained by replacing the first column in $\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{b}^F_\epsilon;\mathbf{q}^F_\epsilon)}}(\bar{\mathbf{a}})$ by the first column of $\mathcal{B}^\epsilon$, so that:

$$\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}}) = \begin{bmatrix} \frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{n_X,\epsilon}} & \frac{\partial \Phi^\epsilon_{m_X+1}}{\partial b_{m_Y+1,\epsilon}} & \cdots & \frac{\partial \Phi^\epsilon_{m_X+1}}{\partial b_{n_Y,\epsilon}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Phi^\epsilon_{n_X}}{\partial q_{1,\epsilon}} & \cdots & 1 & \frac{\partial \Phi^\epsilon_{n_X}}{\partial b_{m_Y+1,\epsilon}} & \cdots & \frac{\partial \Phi^\epsilon_{n_X}}{\partial b_{n_Y,\epsilon}} \\ \frac{\partial \Psi^\epsilon_{m_Y+1}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Psi^\epsilon_{m_Y+1}}{\partial q_{n_X,\epsilon}} & 1 & \cdots & \frac{\partial \Psi^\epsilon_{m_Y+1}}{\partial b_{n_X,\epsilon}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Psi^\epsilon_{n_Y}}{\partial q_{1,\epsilon}} & \cdots & \frac{\partial \Psi^\epsilon_{n_Y}}{\partial q_{n_X,\epsilon}} & \frac{\partial \Psi^\epsilon_{n_Y}}{\partial b_{m_Y+1,\epsilon}} & \cdots & 1 \end{bmatrix} \tag{D2}$$

Note (D1) is well-defined, as from Lemma 1, we have $\left|\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right| \neq 0$. Let $\frac{\partial \Phi^\epsilon_i}{\partial q_{1,\epsilon}} = 0$, $i = m_X + 1, ..., n_X$, and $\frac{\partial \Psi^\epsilon_j}{\partial q_{1,\epsilon}} = 0$, $j = m_Y + 1, ..., n_Y$, in (D2). The matrices $\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})$ and $\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})$ have common terms: the off-diagonal terms of the matrix $\mathcal{B}^\epsilon$ coincide with the off-diagonal terms of the matrix $\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})$ as $Q \equiv \sum_{i \in T_1} q_i$ and $B \equiv \sum_{j \in T_2} b_j$. If $\frac{\partial q_{m_X+1,\epsilon}}{\partial q_{1,\epsilon}} < -1$, then

$$\left|\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right| > \left|\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right|. \tag{D3}$$

Expansion by cofactors of the both sides of (D3), and cancellation among common terms on both sides, lead to:

$$\frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{1,\epsilon}} \left|\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right| > \left|\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right|, \tag{D4}$$

where $\left|\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right|$ stands for the principal minor of order $(n_X - m_X + n_Y - m_Y - 1).(1,1)$ of $\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})$. From (D3), we have $\frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{1,\epsilon}} > 1$. A contradiction as $\frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{1,\epsilon}} < -1$. Then, we have $-\frac{\left|\mathcal{J}'_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right|}{\left|\mathcal{J}_{\boldsymbol{\Upsilon}^\epsilon_{(\mathbf{q}^F_\epsilon;\mathbf{b}^F_\epsilon)}}(\bar{\mathbf{a}})\right|} \leqslant 1$, so $\frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{1,\epsilon}} \geqslant -1$. Next, if $\frac{\partial q_{m_X+1,\epsilon}}{\partial q_{1,\epsilon}} > 1$, then $\frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{1,\epsilon}} < -1$. A contradiction. Then, $\frac{\partial \Phi^\epsilon_{m_X+1}}{\partial q_{1,\epsilon}} < 1$.

As the same holds for all best responses, and for every $i = 1, ..., m_X$, then

$$-\mathbf{I} \leq \frac{\partial[\sigma^\epsilon_{m_X+1}(\mathbf{b}^L_\epsilon;\mathbf{q}^L_\epsilon;\epsilon), ..., \sigma^\epsilon_{n_X}(\mathbf{b}^{\mathbf{L}}_{\boldsymbol{\epsilon}};\mathbf{q}^{\mathbf{L}}_{\boldsymbol{\epsilon}};\boldsymbol{\epsilon})]}{\partial[q_{1,\epsilon}, ..., q_{m_X,\epsilon}]} << \mathbf{I}, \tag{D5}$$

where $\mathbf{I}$ is the $(n_X - m_X, m_X)$ unit matrix.

Now, we show $\frac{\partial \varphi_j^\epsilon(.)}{\partial q_{i,\epsilon}} \geqslant 0$, $j = m_Y+1, ..., n_Y$, $i = 1, ..., m_X$. Assume $\frac{\partial \varphi_j^\epsilon(.)}{\partial q_{i,\epsilon}} < 0$ (strategies are substitutes so goods are complements). Then, $u(.)$ is not differentiable, which contradicts Assumption 2a. Then, the conclusion follows.

6.5. Appendix E: Proof of Proposition 4

Consider a leader of type X (the same holds for a type Y leader). We show that, for each $i = 1, ..., m_X$, the best reply $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$ *exists* and is a continuous *function* (a similar reasoning holds for $\psi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_{-j,\epsilon}^L; \epsilon)$, for each $j = 1, ..., m_Y$). To this end, we study the properties of the reduced form payoff $\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, .; \mathbf{b}_\epsilon^L, .; \epsilon) \equiv \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, \boldsymbol{\sigma}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon))$. Then, we characterize the solution with the Kuhn-Tucker conditions. For all admissible strategy profiles of other leaders, the problem for leader i may be written:

$$\max_{\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)} \{\pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, .; \mathbf{b}_\epsilon^L, .; \epsilon) : q_{i,\epsilon} \in \mathcal{S}_i\},\ \epsilon > 0, \tag{E1}$$

The solution, if it exists, is leader i's best reply (see Definition 3), i.e. $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_{-i}^L; \mathbf{b}^L; \epsilon)$. The set $\mathcal{S}_i = [0, \alpha_i]$ is nonempty, compact and convex. As $u_i(.) \in \mathcal{C}^2$ and $\boldsymbol{\sigma}^\epsilon(.) \in \mathcal{C}^2$, and $\boldsymbol{\varphi}^\epsilon(.) \in \mathcal{C}^2$, we have $\pi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) \in \mathcal{C}^2$. Then, $\pi_i^\epsilon(.)$ is a continuous function of $(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}; \mathbf{b}_\epsilon; \epsilon)$. We show $\pi_i^\epsilon(.)$ is a strictly quasi-concave function of $q_{i,\epsilon}$. Differentiating (6) with respect to $q_{i,\epsilon}$, and using $\boldsymbol{\sigma}^\epsilon(.)$ and $\boldsymbol{\varphi}^\epsilon(.)$ yield:

$$\frac{\partial \pi_i^\epsilon}{\partial q_{i,\epsilon}} = -\frac{\partial u_i}{\partial x_i} + \chi p_X^\epsilon \frac{\partial u_i}{\partial y_i}, \tag{E2}$$

where $\chi \equiv 1 - (1 + \nu_{i,\epsilon}^X)\frac{q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon} + \eta_{i,\epsilon}^X \frac{q_{i,\epsilon}}{\mathbf{B}_\epsilon + \epsilon}$, $\nu_{i,\epsilon}^X = \frac{\partial \sum_i \sigma_i^\epsilon(.)}{\partial q_{i,\epsilon}}$, $\eta_{i,\epsilon}^X = \frac{\partial \sum_j \varphi_j^\epsilon(.)}{\partial q_{i,\epsilon}}$, and $p_X^\epsilon \triangleq p_X(\mathbf{q}_\epsilon^L, \boldsymbol{\sigma}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon))$. By construction $\nu_{i,\epsilon}^X = \nu_\epsilon^X$, and $\eta_{i,\epsilon}^X = \eta_\epsilon^X$, with $\nu_\epsilon^X \in [-1, 1)$ and $\eta_\epsilon^X \geqslant 0$ (from 2a.). Indeed, as $\chi \in [0,1]$, then $0 \leqslant -(1+\nu_\epsilon^X)\frac{q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon} + \eta_\epsilon^X \frac{q_{i,\epsilon}}{\mathbf{B}_\epsilon + \epsilon} \leqslant 1$, which leads to $\frac{\eta_\epsilon^X}{2 - \nu_\epsilon^X} \leqslant \frac{\mathbf{B}_\epsilon + \epsilon}{\mathbf{Q}_\epsilon + \epsilon} \leqslant \frac{\eta_\epsilon^X}{1 + \nu_\epsilon^X}$. Then, $\nu_\epsilon^X \leqslant \frac{1}{2}$. And, from Proposition 3, we get $\nu_\epsilon^X \geqslant -1$ as $\frac{\partial \sigma_i^\epsilon(.)}{\partial q_{i,\epsilon}} \geqslant -1$. Next, by differentiating (E2) with respect to $q_{i,\epsilon}$ leads to:

$$\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} = \frac{\partial^2 u_i}{(\partial x_i)^2} - 2\chi p_X^\epsilon \frac{\partial u_i}{\partial x_i \partial y_i} + (\chi p_X^\epsilon)^2 \frac{\partial^2 u_i}{(\partial y_i)^2} - \kappa \frac{\partial u_i}{\partial y_i}, \tag{E3}$$

where $\kappa \equiv \frac{(1+\nu_\epsilon^X)(B_\epsilon + \epsilon)}{(Q_\epsilon + \epsilon)^2}(2 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{Q_\epsilon + \epsilon}) - \frac{2\eta_\epsilon^X}{Q_\epsilon + \epsilon}(1 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{Q_\epsilon + \epsilon})$. The first three terms on the right hand side of (E3) are equal to the negative of the determinant of the bordered Hessian matrix of u_i, which is positive from (2c). For (E3) to be strictly negative, it is sufficient that $\kappa > 0$, that is, $\frac{B_\epsilon + \epsilon}{Q_\epsilon + \epsilon} > \frac{2\eta_\epsilon^X}{1+\nu_\epsilon^X} \frac{1 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon}}{2 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon}}$. But, as $\chi \leqslant 1$, then $\frac{B_\epsilon + \epsilon}{Q_\epsilon + \epsilon} \leqslant \frac{\eta_\epsilon^X}{1 + \nu_\epsilon^X}$. Assume that $\frac{\eta_\epsilon^X}{1+\nu_\epsilon^X} < \frac{2\eta_\epsilon^X}{1+\nu_\epsilon^X} \frac{1 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon}}{2 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon}}$. Then, $\frac{1}{2} < \frac{1 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon}}{2 - \frac{(1+\nu_\epsilon^X)q_{i,\epsilon}}{\mathbf{Q}_\epsilon + \epsilon}}$. A contradiction. Then, $\kappa > 0$. As $\frac{\partial^2 \pi_i^\epsilon}{(\partial q_{i,\epsilon})^2} < 0$, the solution to (E1) is unique, so $\phi_i^\epsilon(\mathbf{q}_{-i}^L; \mathbf{b}^L; \epsilon)$ is *point-valued*. Then, the mapping $\phi_i^\epsilon(\mathbf{q}_{-i}^L; \mathbf{b}^L; \epsilon)$

is a function. As $\pi_i^\epsilon(.)$ is strictly quasi-concave, and $\mathcal{S}_i$ is quasi-convex, the Kuhn-Tucker conditions are sufficient to identify the solution to (E1). Let

$$\mathcal{L}_i^\epsilon(.;\epsilon) := \pi_i^\epsilon(q_{i,\epsilon}, \mathbf{q}_{-i,\epsilon}^L, \boldsymbol{\sigma}^\epsilon(q_{i,\epsilon},.;\epsilon); \mathbf{b}_\epsilon^L, \boldsymbol{\varphi}^\epsilon(q_{i,\epsilon},.;\epsilon)) + \lambda_{i,\epsilon}(\alpha_i - q_{i,\epsilon}) + \mu_{i,\epsilon} q_{i,\epsilon}, \quad \text{(E4)}$$

where $\lambda_{i,\epsilon}, \mu_{i,\epsilon} \geqslant 0$, $i = 1, ..., m_X$. Then, $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon)$ is the unique solution to:

$$\max_{\phi_i^\epsilon(.)} \mathcal{L}_i^\epsilon(.;\epsilon) = u_i(\alpha_i - q_{i,\epsilon}, \frac{B_\epsilon^L + \sum_j \varphi_j^\epsilon(.;\epsilon) + \epsilon}{q_{i,\epsilon} + Q_{-i,\epsilon}^L + \sum\limits_{k,k\neq i} \sigma_k^\epsilon(.;\epsilon) + \epsilon} q_{i,\epsilon}) + \lambda_{i,\epsilon}(\alpha_i - q_{i,\epsilon}) + \mu_{i,\epsilon} q_{i,\epsilon}. \quad \text{(E5)}$$

For all $\epsilon > 0$, the Kuhn-Tucker conditions may be written:

$$\frac{\partial \mathcal{L}_i^\epsilon}{\partial q_{i,\epsilon}} = -\frac{\partial u_i}{\partial x_i} + \chi p_X^\epsilon \frac{\partial u_i}{\partial y_i} - \lambda_{i,\epsilon} + \mu_{i,\epsilon} = 0, \quad \text{(E6)}$$

$$\lambda_{i,\epsilon} \geqslant 0, \ (\alpha_i - b_{i,\epsilon}) \geqslant 0, \text{ with } \lambda_{i,\epsilon}(\alpha_i - b_{i,\epsilon}) = 0, \quad \text{(E7)}$$

$$\mu_{i,\epsilon} \geqslant 0, \ b_{i,\epsilon} \geqslant 0, \text{ with } \mu_{i,\epsilon}\ b_{i,\epsilon} = 0. \quad \text{(E8)}$$

If $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) > 0$, then $\mu_{i,\epsilon} = 0$, where $b_{i,\epsilon}$ is the solution to:

$$-\frac{\partial u_i}{\partial x_i} + \chi p_X^\epsilon \frac{\partial u_i}{\partial y_i} = \mu_{i,\epsilon}. \quad \text{(E9)}$$

If $\lambda_{i,\epsilon} > 0$, then $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) = \alpha_i$; if $\lambda_{i,\epsilon} = 0$, then $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) \in (0, \alpha_i)$. Now, if $\mu_{i,\epsilon} > 0$, then $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) = 0$ and $\lambda_{i,\epsilon} = 0$ since $q_{i,\epsilon} < \alpha_i$. Then, either $\phi_i^\epsilon(\mathbf{b}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^L; \epsilon) > 0$ when $b_{i,\epsilon} \in (0, \alpha_i]$ or $\phi_i^\epsilon(\mathbf{b}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^L; \epsilon) = 0$. Then, there is a unique maximum $q_{i,\epsilon} = \phi_i^\epsilon(\mathbf{b}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^L; \epsilon) \geqslant 0$, $i = 1, ..., m_X$.

Finally, by using Berge Maximum Theorem, $\phi_i^\epsilon(\mathbf{q}_{-i,\epsilon}^L; \mathbf{b}_\epsilon^L; \epsilon) \in \mathcal{C}^0$, $i = 1, ..., m_X$.

6.6. Appendix F: Proof of Lemma 3

To show Lemma 3, we adapt to our sequential framework one result based on the Uniform Monotonicity Lemma (see Lemma C, p. 8, in Dubey and Shubik, 1978).

LEMMA 6 (Uniform monotonicity). *Let $c \in \{X, Y\}$, let $u_k : \mathbb{R}_+^2 \to \mathbb{R}$, $z_k \longmapsto u_k(z_k)$, $k = i, j$, $i \in T_X$, $j \in T_Y$, be a continuous and increasing function, and let H be a positive constant. Then, there exists a positive real number $h(u_k(.), c, H) \in (0, 1)$ such that, for all $s_k, z_k \in \mathbb{R}_+^2$, if $\|\mathbf{z}_k\| \leqslant H$ and $\|\mathbf{s}_k - \mathbf{z}_k\| \leqslant h(u_k(.), c, H)$, then $u_k(s_k + e^c) > u_k(z_k)$, where $\|.\|$ denotes the Euclidean norm, and e^c denotes the vector in $\mathbb{R}_+^2$ whose c-th component is 1 and the other 0.*

PROOF. Lemma 6 is a direct consequence of Lemma C in Dubey and Shubik (1978) (see Appendix B, p. 19) as, for each k, $u_k(.)$ satisfies Assumptions 2a-2b.

We now show that there exist some uniform bounds on the relative price in each perturbed subgame. First, we show there is $\xi_1 > 0$ such that $p_X^\epsilon > \xi_1$. Second, we show there is $\xi_2 > 0$ such that $p_X^\epsilon < \xi_2$. Let $(\tilde{\mathbf{q}}_\epsilon^L, \tilde{\mathbf{q}}_\epsilon^F(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L); \tilde{\mathbf{b}}_\epsilon^L, \tilde{\mathbf{b}}_\epsilon^F(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L))$ be an ϵ-SNE, and let $\tilde{p}_X^\epsilon \triangleq p_X^\epsilon(\tilde{\mathbf{q}}_\epsilon^L, \tilde{\mathbf{q}}_\epsilon^F(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L); \tilde{\mathbf{b}}_\epsilon^L, \tilde{\mathbf{b}}_\epsilon^F(\tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L))$ be the corresponding relative price.

1. First, we show the existence of $\xi_1 > 0$ such that $\tilde{p}_X^\epsilon > \xi_1$. Consider one leader j and one follower j'. Let

$$\begin{aligned} H &= \max\left\{\bar{\alpha}, \bar{\beta}\right\}, \text{ with } \bar{\alpha} \equiv \sum_{i\in T_X} \alpha_i \text{ and } \bar{\beta} \equiv \sum_{j\in T_Y} \beta_j; \\ h &= \min\{h(u_j, Y, H), h(u_{j'}, Y, H)\}; \\ A &= \frac{1}{2}\min\{\beta_j, \beta_{j'}\},\ j \neq j'. \end{aligned} \tag{F1}$$

Assume, without loss of generality, that $\tilde{b}_{j,\epsilon} \leqslant \frac{\tilde{B}_\epsilon}{2}$ or $\tilde{b}_{j',\epsilon} \leqslant \frac{\tilde{B}_\epsilon}{2}$, for at least one leader j or one follower j' (otherwise $\tilde{b}_{j,\epsilon}+ \tilde{b}_{j',\epsilon} > \tilde{B}_\epsilon$). Consider an increase of strategic supply at each stage.

Consider first follower j'. Suppose $\beta_{j'} - \tilde{b}_{j',\epsilon} \geqslant A$. Then, an increase δ in follower j''s supply such that $b_{j',\epsilon}(\delta) = \tilde{b}_{j',\epsilon} + \delta$, with $\delta \in (0, \frac{1}{2}\min\{\epsilon, A\}]$, has the following incremental effect on his final holding:

$$\begin{aligned} x_{j',\epsilon}(\delta) - x_{j',\epsilon} &= \frac{\tilde{Q}_\epsilon + \epsilon}{\tilde{B}_\epsilon + \epsilon + \delta}(\tilde{b}_{j',\epsilon} + \delta) - \frac{\tilde{Q}_\epsilon + \epsilon}{\tilde{B}_\epsilon + \epsilon}\tilde{b}_{j',\epsilon} \\ &= \delta\frac{\tilde{B}_\epsilon + \epsilon - \tilde{b}_{j',\epsilon}}{\tilde{B}_\epsilon + \epsilon + \delta}\frac{\tilde{Q}_\epsilon + \epsilon}{\tilde{B}_\epsilon + \epsilon} \\ &> \delta\frac{\frac{\tilde{B}_\epsilon}{2} + \frac{\epsilon}{2} + \frac{\delta}{2}}{\tilde{B}_\epsilon + \epsilon + \delta}\frac{\tilde{Q}_\epsilon + \epsilon}{\tilde{B}_\epsilon + \epsilon} = \frac{\delta}{2}\frac{1}{\tilde{p}_X^\epsilon}, \end{aligned} \tag{F2}$$

and

$$y_{j',\epsilon}(\delta) - y_{j',\epsilon} = (\beta_{j'} - \tilde{q}_{j',\epsilon} - \delta) - (\beta_{j'} - \tilde{q}_{j',\epsilon}) = -\delta, \tag{F3}$$

where the strict inequality in (F2) results from $\tilde{B}_\epsilon + \epsilon - \tilde{b}_{j',\epsilon} \geqslant \frac{\tilde{B}_\epsilon}{2} + \epsilon > \frac{\tilde{B}_\epsilon}{2} + \frac{\epsilon}{2} + \frac{\delta}{2}$ (as $\tilde{b}_{j',\epsilon} \leqslant \frac{\tilde{B}_\epsilon}{2}$ and $\delta \leqslant \frac{1}{2}\epsilon$). Let us define

$$t = -2\tilde{p}_X^\epsilon \mathbf{e}^Y, \text{ where } \mathbf{e}^Y = (0, 1). \tag{F4}$$

Then, the following vector inequality holds:

$$\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}(\delta), p_X^\epsilon(\tilde{\mathbf{q}}_\epsilon^L, \tilde{\mathbf{q}}_\epsilon^F(.); \tilde{\mathbf{b}}_\epsilon^L, b_{j',\epsilon}(\delta), \tilde{\mathbf{b}}_{-j,\epsilon}^F(.))) \geq \mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon) + \frac{\delta}{2}\frac{1}{\tilde{p}_X^\epsilon}(\mathbf{e}^X + t), \tag{F5}$$

where $\mathbf{e}^X = (1, 0)$, and where, by (F2), the inequality (F5) is strict for the first component of $\mathbf{z}_{j',\epsilon}$. We can now apply Lemma 6, with $c = X$, $\mathbf{z}_{j',\epsilon} = \mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon)$ and $\mathbf{s}_{j',\epsilon} = \mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon) + t$. We know that $\mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon) \in \mathbb{R}_+^2$ and $\left\|\mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon)\right\| \leqslant H$. If $\mathbf{s}_{j',\epsilon} \in \mathbb{R}_+^2$ and $\|t\| \leqslant h$, then, by Lemma 6, we get:

$$u_{j'}(\mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon) + \mathbf{e}^X + t) > u_{j'}(\mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon)). \tag{F6}$$

As $u_{j'}$ satisfies Assumptions (2b) and (2c), and as $0 < \frac{\delta}{2}\frac{1}{\tilde{p}_X^\epsilon} < 1$, then we deduce:

$$u_{j'}(\mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon) + \frac{\delta}{2}\frac{1}{\tilde{p}_X^\epsilon}(\mathbf{e}^X + t)) > u_{j'}(\mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon)). \tag{F7}$$

Then, as (F5) holds strictly for its first component, from (2b), we deduce:

$$u_{j'}(\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}(\delta), p_X^\epsilon(\tilde{\mathbf{q}}_\epsilon^L, \tilde{\mathbf{q}}_\epsilon^F(.); \tilde{\mathbf{b}}_\epsilon^L, b_{j',\epsilon}(\delta), \tilde{\mathbf{b}}_{-j,\epsilon}^F(.)))) > u_{j'}(\mathbf{z}_{j',\epsilon}(\tilde{b}_{j',\epsilon}, \tilde{p}_X^\epsilon)), \quad \text{(F8)}$$

a contradiction. Hence, either $\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}_X^\epsilon)+t < \mathbf{0}$ or $\|t\| > h$. If $\mathbf{z}_{j',\epsilon}(b_{j',\epsilon}, \tilde{p}_X^\epsilon)+t < \mathbf{0}$, then, $\tilde{y}_{j',\epsilon} - 2\tilde{p}_X^\epsilon < 0$. As $\tilde{y}_{j',\epsilon} = \beta_{j'} - \tilde{b}_{j',\epsilon} \geqslant A$, we deduce:

$$\tilde{p}_X^\epsilon > \frac{A}{2}. \quad \text{(F9)}$$

Suppose now we have $\|t\| > h$. Then, we deduce:

$$\tilde{p}_X^\epsilon > \frac{h}{2}. \quad \text{(F10)}$$

Finally, assume that the inequality $\beta_{j'} - \tilde{b}_{j',\epsilon} \geqslant A$ does not hold, which means that $\beta_{j'} - \tilde{b}_{j',\epsilon} < A$. Then, we have $\tilde{b}_{j',\epsilon} > \beta_j - A \geqslant A$. Then $\tilde{b}_{j',\epsilon} > A$, so, we get:

$$\tilde{p}_X^\epsilon > \frac{A}{\bar{\alpha}}. \quad \text{(F11)}$$

Therefore, it suffices to take for follower j':

$$\xi_1^{j'} = \min\left\{\frac{A}{2}, \frac{h}{2}, \frac{A}{\bar{\alpha}}\right\}. \quad \text{(F12)}$$

Consider now leader j. Assume $\beta_j - \tilde{b}_{j,\epsilon} \geqslant A$. Let $b_{j,\epsilon}(\delta) = \tilde{b}_{j,\epsilon} + \delta$, with $\delta \in (0, \frac{1}{2}\min\{\epsilon, A\}]$. As $\tilde{p}_X^\epsilon = \frac{\sum_{i=1}^{m_X} \tilde{q}_{i,\epsilon} + \sum_j \varphi_j^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L) + \epsilon}{\sum_{j=1}^{m_Y} \tilde{b}_{j,\epsilon} + \sum_i \sigma_i^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L) + \epsilon}$, then we have:

$$\begin{aligned}
x_{j,\epsilon}(\delta) - x_{j,\epsilon} &= \frac{\sum_{i=1}^{m_X} \tilde{q}_{i,\epsilon} + \sum_i \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L + \delta) + \epsilon}{\sum_{j=1}^{m_Y} \tilde{b}_{j,\epsilon} + \delta + \sum_j \varphi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L + \delta) + \epsilon}(\tilde{b}_{j,\epsilon} + \delta) - \frac{\tilde{b}_{j,\epsilon}}{\tilde{p}_X^\epsilon} \quad \text{(F13)} \\
&= \delta \frac{\tilde{B}_\epsilon + \epsilon - (1+\nu_\epsilon^Y)\tilde{b}_{j,\epsilon}}{\tilde{B}_\epsilon + \epsilon + (1+\nu_\epsilon^Y)\delta} \frac{\tilde{Q}_\epsilon + \epsilon}{\tilde{B}_\epsilon + \epsilon} + \delta\eta_\epsilon^Y \frac{\tilde{b}_{j,\epsilon} + \delta}{\tilde{B}_\epsilon + \epsilon + (1+\nu_\epsilon^Y)\delta} \\
&> \delta(1-\nu_\epsilon^Y) \frac{\frac{\tilde{\mathbf{B}}_\epsilon}{2} + \frac{\epsilon}{2} + (1+\nu_\epsilon^Y)\frac{\delta}{2}}{\tilde{B}_\epsilon + \epsilon + (1+\nu_\epsilon^Y)\delta} \frac{1}{\tilde{p}_X^\epsilon} + \delta a \eta_\epsilon^Y \\
&= \frac{\delta}{2}\left(\frac{1-\nu_\epsilon^Y}{\tilde{p}_X^\epsilon} + 2a\eta_\epsilon^Y\right),
\end{aligned}$$

and

$$y_{j,\epsilon}(\delta) - y_{j,\epsilon} = -\delta, \quad \text{(F14)}$$

where $a \equiv \frac{\tilde{b}_{j,\epsilon} + \delta}{\tilde{B}_\epsilon + \epsilon + (1+\nu_\epsilon^Y)\delta}$, with $0 < a \leqslant 1$, $\nu_\epsilon^Y = \frac{\partial \sum_j \varphi_j^\epsilon(.)}{\partial b_{j,\epsilon}}$ and $\eta_\epsilon^Y = \frac{\partial \sum_i \sigma_i^\epsilon(.)}{\partial b_{j,\epsilon}}$, and where the strict inequality results from $\tilde{B}_\epsilon + \epsilon - (1+\nu_\epsilon^Y)\tilde{b}_{j,\epsilon} \geqslant (1-\nu_\epsilon^Y)\frac{\tilde{B}_\epsilon}{2} + \epsilon > (1-\nu_\epsilon^Y)(\frac{\tilde{B}_\epsilon}{2} + \frac{\epsilon}{2} + (1+\nu_\epsilon^Y)\frac{\delta}{2})$ as $\tilde{b}_{j,\epsilon} \leqslant \frac{\tilde{B}_\epsilon}{2}$, $\delta \leqslant \frac{1}{2}\epsilon$ and $\nu_\epsilon^Y \in [-1, \frac{1}{2}]$. Let us define

$$t = -2\frac{\tilde{p}_X^\epsilon}{1 - \nu_\epsilon^Y + 2a\eta_\epsilon^Y \tilde{p}_X^\epsilon}\mathbf{e}^Y. \quad \text{(F15)}$$

Then, the following vector inequality holds:

$$\mathbf{z}_{j,\epsilon}(b_{j,\epsilon}(\delta), p^{\epsilon}_X(\tilde{\mathbf{q}}^L_{\epsilon}; \mathbf{q}^F_{\epsilon}(\tilde{\mathbf{q}}^L_{\epsilon}; b_{j,\epsilon}(\delta), \tilde{\mathbf{b}}^L_{-j,\epsilon}); b_{j,\epsilon}(\delta), \tilde{\mathbf{b}}^L_{-j,\epsilon}, \mathbf{b}^F_{\epsilon}(\tilde{\mathbf{q}}^L_{\epsilon}; b_{j,\epsilon}(\delta), \tilde{\mathbf{b}}^L_{-j,\epsilon}))) \geq \\ \mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X) + \tfrac{\delta}{2}\tfrac{1-\nu^Y_{\epsilon}+2a\eta^Y_{\epsilon}\tilde{p}^{\epsilon}_X}{\tilde{p}^{\epsilon}_X}(\mathbf{e}^X + t). \tag{F16}$$

Let $c = X$, $\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X)$ and $\mathbf{s}_{j,\epsilon} = \mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X) + t$. We know that $\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X) \in \mathbb{R}^2_+$ and $\left\|\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X)\right\| \leqslant H$. Suppose that $\mathbf{s}_{j,\epsilon} \in \mathbb{R}^2_+$ and $\|t\| \leqslant h$. Then, by Lemma 6, we deduce:

$$u_j(\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X) + \mathbf{e}^X + t) > u_j(\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X)). \tag{F17}$$

From Assumptions (2b) and (2c) and as $0 < \delta(\frac{1}{2}\frac{1-\nu^Y_{\epsilon}}{\tilde{p}^{\epsilon}_X} + a\eta^Y_{\epsilon}) < 1$, we deduce:

$$u_j(\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X) + \frac{\delta}{2}\frac{1-\nu^Y_{\epsilon}+2a\eta^Y_{\epsilon}\tilde{p}^{\epsilon}_X}{\tilde{p}^{\epsilon}_X}(\mathbf{e}^X + t)) > u_j(\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X)). \tag{F18}$$

But then, by Assumptions (2b) and (2c), we deduce:

$$u_j(\mathbf{z}_{j,\epsilon}(b_{j,\epsilon}(\delta), p^{\epsilon}_X(\tilde{\mathbf{q}}^L_{\epsilon}; \mathbf{q}^F_{\epsilon}(\tilde{\mathbf{q}}^L_{\epsilon}; b_{j,\epsilon}(\delta), \tilde{\mathbf{b}}^L_{-j,\epsilon}); b_{j,\epsilon}(\delta), \tilde{\mathbf{b}}^L_{-j,\epsilon}, \mathbf{b}^F_{\epsilon}(\tilde{\mathbf{q}}^L_{\epsilon}; b_{j,\epsilon}(\delta), \tilde{\mathbf{b}}^L_{-j,\epsilon})))) > \\ u_j(\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X)), \tag{F19}$$

a contradiction. Hence, either $\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X) + t < \mathbf{0}$ or $\|t\| > h$. Thus, if $\mathbf{z}_{j,\epsilon}(\tilde{b}_{j,\epsilon}, \tilde{p}^{\epsilon}_X) + t < \mathbf{0}$, then, $\tilde{y}_{j,\epsilon} - \frac{2\tilde{p}^{\epsilon}_X}{1-\nu^Y_{\epsilon}+2a\eta^Y_{\epsilon}\tilde{p}^{\epsilon}_X} < 0$. Then, we have:

$$\tilde{p}^{\epsilon}_X > \frac{A}{2}\left(\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}A}\right), \tag{F20}$$

as $\tilde{y}_{j,\epsilon} = \beta_j - \tilde{b}_{j,\epsilon} \geqslant A$, where $\frac{A}{2}\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}A} > 0$. Reason: $\frac{A}{2}\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}A} \geqslant \frac{A}{2}(1-\nu^Y_{\epsilon}) > 0$. The strict inequality holds as $\frac{A}{2} > 0$ and $\nu^Y_{\epsilon} < 1$, while the weak inequality results from $a\eta^Y_{\epsilon}A \geqslant 0$ since $0 < a \leqslant 1$, $A > 0$, and $\eta^Y_{\epsilon} \geqslant 0$ (remind, from (2a), that $u_{j'}$ is differentiable so η^Y_{ϵ} cannot be negative, and $\chi \in [-1, 1]$ in (E2)). Next, if $\|t\| > h$, then:

$$\tilde{p}^{\epsilon}_X > \frac{h}{2}\left(\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}h}\right), \tag{F21}$$

where $\frac{h}{2}\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}h} > 0$. Reason: $\frac{h}{2}\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}h} \geqslant \frac{h}{2}(1-\nu^Y_{\epsilon}) > 0$. The strict inequality holds as $\frac{h}{2} \in (0, \frac{1}{2})$ and $\nu^Y_{\epsilon} < 1$, while the weak inequality results from $a\eta^Y_{\epsilon}h \geqslant 0$ since $0 < a \leqslant 1$, $h \in (0, 1)$, and $\eta^Y_{\epsilon} \geqslant 0$. Finally, assume that $\beta_j - \tilde{b}_{j,\epsilon} \geqslant A$ does not hold, i.e. $\beta_j - \tilde{b}_{j,\epsilon} < A$. Then, $\tilde{b}_{j,\epsilon} > \beta_j - A \geqslant A$. Then, $\tilde{b}_{j,\epsilon} > A$, so we deduce:

$$\tilde{p}^{\epsilon}_X > \frac{A}{\bar{\alpha}}. \tag{F22}$$

Therefore, it suffices to take for leader j:

$$\xi^j_1 = \min\left\{\frac{A}{2}\left(\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}A}\right), \frac{h}{2}\left(\frac{1-\nu^Y_{\epsilon}}{1-a\eta^Y_{\epsilon}h}\right), \frac{A}{\bar{\alpha}}\right\}. \tag{F23}$$

Then, by taking $\xi_1 = \min(\xi_1^j, \xi_1^{j'})$, where $\xi_1 > 0$, we conclude that:

$$\tilde{p}_X^\epsilon > \xi_1. \tag{F24}$$

2. Second, we show there is $\xi_2 > 0$ such that $\tilde{p}_X^\epsilon < \xi_2$. Consider one leader i and one follower i'. Let

$$\begin{aligned} \hat{h} &= \min\{h(u_j, Y, H), h(u_{j'}, Y, H)\}; \\ \hat{A} &= \frac{1}{2}\min\{\alpha_i, \alpha_{i'}\},\ i \neq i'. \end{aligned} \tag{F25}$$

Assume $\tilde{q}_{i,\epsilon} \leqslant \frac{\tilde{Q}_\epsilon}{2}$ or $\tilde{q}_{i',\epsilon} \leqslant \frac{\tilde{Q}_\epsilon}{2}$. Consider follower i'. Assume $\alpha_{i'} - \tilde{q}_{i',\epsilon} \geqslant \hat{A}$. Let $q_{i',\epsilon}(\delta) = \tilde{q}_{i',\epsilon} + \delta$, with $\delta \in (0, \frac{1}{2}\min\{\epsilon, \hat{A}\}]$. Then, his final holding may be written:

$$x_{i',\epsilon}(\delta) - x_{i',\epsilon} = -\delta; \tag{F26}$$

and

$$y_{i',\epsilon}(\delta) - y_{i',\epsilon} > \frac{\delta}{2}\tilde{p}_X^\epsilon, \tag{F27}$$

as $\tilde{Q}_\epsilon + \epsilon - \tilde{q}_{i',\epsilon} \geqslant \frac{\tilde{Q}_\epsilon}{2} + \epsilon \geqslant \frac{\tilde{Q}_\epsilon + \epsilon - \tilde{q}_{i',\epsilon}}{2} + \frac{\epsilon}{2} + \frac{\delta}{2}$ (as $\delta < \epsilon$). Let us define

$$t = -\frac{2}{\tilde{p}_X^\epsilon}\mathbf{e}^X. \tag{F28}$$

Then, we have the vector inequality:

$$\mathbf{z}_{i',\epsilon}(q_{i',\epsilon}(\delta), p_X^\epsilon(\tilde{\mathbf{q}}_\epsilon^L, q_{i',\epsilon}(\delta), \tilde{\mathbf{q}}_{-i',\epsilon}^F(.); \tilde{\mathbf{b}}_\epsilon^L, \tilde{\mathbf{b}}_\epsilon^F(.))) \geq \mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, \tilde{p}_X^\epsilon) + \frac{\delta}{2}\tilde{p}_X^\epsilon(t + \mathbf{e}^Y). \tag{F29}$$

Suppose that $\mathbf{r}_{i,\epsilon} \in \mathbb{R}_+^2$ and $\|t\| \leqslant h$. Then, by Lemma 6, with $c = Y$, we get:

$$u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, \tilde{p}_X^\epsilon) + t + \mathbf{e}^Y) > u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, \tilde{p}_X^\epsilon)). \tag{F30}$$

As Assumptions (2b) and (2c) hold for $u_{i'}$, and as $0 < \frac{\delta}{2}\tilde{p}^\epsilon < 1$, then:

$$u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, \tilde{p}_X^\epsilon) + \frac{\delta}{2}\tilde{p}_X^\epsilon(t + \mathbf{e}^Y)) > u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, \tilde{p}_X^\epsilon)). \tag{F31}$$

But then, by Assumptions (2b) and (2c), we have that:

$$u_{i'}(\mathbf{z}_{i',\epsilon}(q_{i',\epsilon}(\delta), p_X^\epsilon(q_{i',\epsilon}(\delta), \tilde{\mathbf{q}}_{-i',\epsilon}; \tilde{\mathbf{b}}_\epsilon))) > u_{i'}(\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, \tilde{p}_X^\epsilon)), \tag{F32}$$

a contradiction. Then, either $\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p_X^\epsilon) + t < \mathbf{0}$ or $\|t\| > h$. Thus, if $\mathbf{z}_{i',\epsilon}(\tilde{q}_{i',\epsilon}, p_X^\epsilon) + t < \mathbf{0}$, then, $\tilde{x}_{i',\epsilon} - \frac{2}{\tilde{p}_X^\epsilon(\tilde{\mathbf{q}}_\epsilon; \tilde{\mathbf{b}}_\epsilon)} < 0$. As $\tilde{x}_{i',\epsilon} = \alpha_{i'} - \tilde{q}_{i',\epsilon} \geqslant \hat{A}$, we deduce:

$$\tilde{p}_X^\epsilon < \frac{2}{\hat{A}}. \tag{F33}$$

Suppose now we have $\|t\| > h$. Then, we deduce:

$$\tilde{p}_X^\epsilon < \frac{2}{\hat{h}}. \tag{F34}$$

Finally, assume $\alpha_i - \tilde{q}_{i',\epsilon} < \hat{A}$. Then, $\tilde{q}_{i',\epsilon} > \alpha_i - \hat{A} \geqslant \hat{A}$, so $\tilde{q}_{i',\epsilon} > \hat{A}$. We deduce:

$$\tilde{p}_X^\epsilon < \frac{\bar{\beta}}{\hat{A}}. \tag{F35}$$

Therefore, it is sufficient to take:

$$\xi_2^{i'} = \max\left\{\frac{2}{\hat{A}}, \frac{2}{\hat{h}}, \frac{\bar{\beta}}{\hat{A}}\right\}. \tag{F36}$$

Consider now leader i. Assume that $\alpha_i - \tilde{q}_{i,\epsilon} \geqslant \hat{A}$. Let $q_{i,\epsilon}(\delta) = \tilde{q}_{i,\epsilon} + \delta$, with $\delta \in (0, \frac{1}{2}\min\{\epsilon, \hat{A}\}]$. Such an increase has the following effect on her final holding:

$$x_{i,\epsilon}(\delta) - x_{i,\epsilon} = -\delta, \tag{F37}$$

and

$$\begin{aligned} y_{i,\epsilon}(\delta) - y_{i,\epsilon} &= \frac{\sum_{j=1}^{m_Y} \tilde{b}_{j,\epsilon} + \sum_j \varphi_j^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L + \delta) + \epsilon}{\sum_{i=1}^{m_X} \tilde{q}_{i,\epsilon} + \delta + \sum_i \sigma_i^\epsilon(\mathbf{b}_\epsilon^L; \mathbf{q}_\epsilon^L + \delta) + \epsilon}(\tilde{q}_{i,\epsilon} + \delta) - \tilde{p}_X^\epsilon \tilde{q}_{i,\epsilon} \quad \text{(F38)} \\ &= \delta \frac{\tilde{Q}_\epsilon + \epsilon - (1+\nu_\epsilon^X)\tilde{q}_{i,\epsilon}}{\tilde{Q}_\epsilon + (1+\nu_\epsilon^X)\delta + \epsilon} \frac{\tilde{B}_\epsilon + \epsilon}{\tilde{Q}_\epsilon + \epsilon} + \delta\eta_\epsilon^X \frac{\tilde{q}_{i,\epsilon} + \delta}{\tilde{Q}_\epsilon + (1+\nu_\epsilon^X)\delta + \epsilon} \\ &> \delta(1-\nu_\epsilon^X) \frac{\frac{\tilde{Q}_\epsilon}{2} + (1+\nu_\epsilon^X)\frac{\delta}{2} + \frac{\epsilon}{2}}{\tilde{Q}_\epsilon + (1+\nu_\epsilon^X)\delta + \epsilon} \tilde{p}_X + \delta d \eta_\epsilon^X \\ &= \frac{\delta}{2}((1-\nu_\epsilon^X)\tilde{p}_X^\epsilon + 2d\eta_\epsilon^X), \end{aligned}$$

where $d \equiv \frac{\tilde{q}_{i,\epsilon}+\delta}{\tilde{Q}_\epsilon + (1+\nu_\epsilon^X)\delta + \epsilon}$, with $0 < d \leqslant 1$, $\nu_\epsilon^X = \frac{\partial \sum_i \sigma_i^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)}{\partial q_{i,\epsilon}}$ and $\eta_\epsilon^X = \frac{\partial \sum_j \varphi_j^\epsilon(\mathbf{q}_\epsilon^L; \mathbf{b}_\epsilon^L; \epsilon)}{\partial q_{i,\epsilon}}$ for δ sufficiently small, and where the strict inequality results from $\tilde{Q}_\epsilon + \epsilon - (1+\nu_\epsilon^X)\tilde{q}_{i,\epsilon} \geqslant (1-\nu_\epsilon^X)(\frac{\tilde{Q}_\epsilon}{2} + \frac{\epsilon}{2} + (1+\nu_\epsilon^X)\frac{\delta}{2})$ as $\tilde{q}_{i,\epsilon} \leqslant \frac{\tilde{Q}_\epsilon}{2}$ always holds, and as $\delta < \epsilon$, with $\nu_\epsilon^X \in [-1, 1)$. Let us define

$$t = -2\frac{1}{(1-\nu_\epsilon^X)\tilde{p}_X^\epsilon + 2d\eta_\epsilon^X}\mathbf{e}^X. \tag{F39}$$

Then, the following vector inequality holds:

$$\begin{gathered} \mathbf{z}_{i,\epsilon}(q_{i,\epsilon}(\delta), p_X^\epsilon(q_{i,\epsilon}(\delta), \tilde{\mathbf{q}}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^F(q_{i,\epsilon}(\delta), \tilde{\mathbf{q}}_\epsilon^L; \tilde{\mathbf{b}}_\epsilon^L); \tilde{\mathbf{b}}_\epsilon^L, \mathbf{b}_\epsilon^F(q_{j,\epsilon}(\delta), \tilde{\mathbf{q}}_{-i,\epsilon}^L; \tilde{\mathbf{b}}_\epsilon^L))) \geq \\ \mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon) + \tfrac{\delta}{2}((1-\nu_\epsilon^X)\tilde{p}_X^\epsilon + 2d\eta_\epsilon^X)(t + \mathbf{e}^Y). \end{gathered} \tag{F40}$$

Suppose that $\mathbf{s}_{i,\epsilon} \in \mathbb{R}_+^2$ and $\|t\| \leqslant h$. Then, by Lemma 6, we deduce:

$$u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon) + t + \mathbf{e}^Y) > u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon)). \tag{F41}$$

From (2b) and (2c) and as $0 < \delta(\frac{1-\nu_\epsilon^X}{2}\tilde{p}_X^\epsilon + d\eta_\epsilon^X) < 1$, we deduce:

$$u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon)) + \frac{\delta}{2}((1-\nu_\epsilon^X)\tilde{p}_X^\epsilon + 2d\eta_\epsilon^X)(t + \mathbf{e}^Y)) > u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon)). \tag{F42}$$

But then, by Assumptions (2b) and (2c), we have that:

$$u_i(q_{i,\epsilon}(\delta), p_X^\epsilon(q_{i,\epsilon}(\delta), \tilde{\mathbf{q}}_{-i,\epsilon}^L; \mathbf{q}_\epsilon^F(q_{i,\epsilon}(\delta), \tilde{\mathbf{q}}_{-i,\epsilon}^L; \tilde{\mathbf{b}}_\epsilon^L); \tilde{\mathbf{b}}_\epsilon^L, \mathbf{b}_\epsilon^F(q_{j,\epsilon}(\delta), \tilde{\mathbf{q}}_{-i,\epsilon}^L; \tilde{\mathbf{b}}_\epsilon^L)))) > \\ u_i(\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon)), \tag{F43}$$

a contradiction. Then, either $\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon) + t < \mathbf{0}$ or $\|t\| > h$. Thus, if $\mathbf{z}_{i,\epsilon}(\tilde{q}_{i,\epsilon}, \tilde{p}_X^\epsilon) + t < \mathbf{0}$, then, $\tilde{x}_{i,\epsilon} - 2\frac{1}{(1-\nu_\epsilon^X)\tilde{p}_X^\epsilon + 2d\eta_\epsilon^X} < 0$. As $\tilde{x}_{i,\epsilon} = \alpha_i - \tilde{q}_{i,\epsilon} \geqslant \hat{A}$, we get:

$$\tilde{p}_X^\epsilon < \frac{2}{\hat{A}}\left(\frac{1 - d\eta_\epsilon^X \hat{A}}{1 - \nu_\epsilon^X}\right), \tag{F44}$$

where $\frac{2}{\hat{A}}\frac{1-d\eta_\epsilon^X\hat{A}}{1-\nu_\epsilon^X} > 0$. Reason: $\frac{2}{\hat{A}}\frac{1-d\eta_\epsilon^X\hat{A}}{1-\nu_\epsilon^X} \geqslant \frac{2}{\hat{A}}\frac{d^2}{1-\nu_\epsilon^X} > 0$ as $d \in (0,1]$ and $\nu_\epsilon^X < 1$. The weak inequality leads to $d^2 + d\eta_\epsilon^X\hat{A} - 1 \leqslant 0$, so $d \leqslant -\frac{\eta_\epsilon^X\hat{A}}{2} + \frac{\sqrt{(\eta_\epsilon^X\hat{A})^2+4}}{2}$, with $0 \leqslant d \leqslant 1$. Then we must have $-\frac{\eta_\epsilon^X\hat{A}}{2} + \frac{\sqrt{(\eta_\epsilon^X\hat{A})^2+4}}{2} \leqslant 1$, which holds as $\eta_\epsilon^X\hat{A} \geqslant 0$.

Next, if $\|t\| > h$, then:

$$\tilde{p}_X^\epsilon < \frac{2}{\hat{h}}\left(\frac{1 - d\eta_\epsilon^X \hat{h}}{1 - \nu_\epsilon^X}\right), \tag{F45}$$

where $\frac{2}{\hat{h}}\frac{1-d\eta_\epsilon^X\hat{h}}{1-\nu_\epsilon^X} > 0$. Reason: $\frac{2}{\hat{h}}\frac{1-d\eta_\epsilon^X\hat{h}}{1-\nu_\epsilon^X} \geqslant \frac{2}{\hat{h}}\frac{d^2}{1-\nu_\epsilon^X} > 0$. The weak inequality leads to $d^2 + d\eta_\epsilon^X\hat{h} - 1 \leqslant 0$, which yields $d \leqslant -\frac{\eta_\epsilon^X\hat{h}}{2} + \frac{\sqrt{(\eta_\epsilon^X\hat{h})^2+4}}{2}$, with $d > 0$. As $d \leqslant 1$, we must have $-\frac{\eta_\epsilon^X\hat{h}}{2} + \frac{\sqrt{(\eta_\epsilon^X\hat{h})^2+4}}{2} \leqslant 1$, which is satisfied as $\eta_\epsilon^X\hat{h} \geqslant 0$.

Finally, assume that the inequality $\alpha_i - \tilde{q}_{i,\epsilon} \geqslant \hat{A}$ does not hold, i.e., $\alpha_i - \tilde{q}_{i,\epsilon} < \hat{A}$. Then, we have $\tilde{q}_{i,\epsilon} > \alpha_i - \hat{A} \geqslant \hat{A}$. Then, we have $\tilde{q}_{i,\epsilon} > \hat{A}$, so we deduce:

$$\tilde{p}_X^\epsilon < \frac{\bar{\beta}}{\hat{A}}. \tag{F46}$$

Therefore, it suffices to take for leader i:

$$\xi_1^i = \max\left\{\frac{2}{\hat{A}}\left(\frac{1 - d\eta_\epsilon^X\hat{A}}{1 - \nu_\epsilon^X}\right), \frac{2}{\hat{h}}\left(\frac{1 - d\eta_\epsilon^X\hat{h}}{1 - \nu_\epsilon^X}\right), \frac{\bar{\beta}}{\hat{A}}\right\}. \tag{F47}$$

Then, by taking $\xi_2 = \max(\xi_2^i, \xi_2^{i'})$, we conclude that:

$$\tilde{p}_X^\epsilon < \xi_2. \tag{F48}$$

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