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Nadia Boussaha
Faycal Hamdi
Saïd Souam

EconomiX - UMR7235
Université Paris Nanterre
Bâtiment G - Maurice Allais, 200, Avenue de la République
92001 Nanterre cedex

Email : secretariat@economix.fr

UMR 7235
Université Paris Nanterre

Multivariate Periodic Stochastic Volatility Models: Applications to Algerian dinar exchange rates and oil prices modeling

Nadia Boussaha[a], Fayçal Hamdi[a,*], Saïd Souam[b]

[a]*RECITS Laboratory, Faculty of Mathematics, USTHB, Algiers, Algeria.*
[b]*Université Paris Nanterre (EconomiX) and CREST.*

Abstract

The contribution of this paper is twofold. In a first step, we propose the so called Periodic Multivariate Autoregressive Stochastic Volatility ($PVAR$-SV) model, that allows the Granger causality in volatility in order to capture periodicity in stochastic conditional variance. After a thorough discussion, we provide some probabilistic properties of this class of models. We thus propose two methods for the estimation problem, one based on the periodic Kalman filter and the other on the particle filter and smoother with Expectation-Maximization (EM) algorithm. In a second step, we propose an empirical application by modeling oil price and three exchange rates time series. It turns out that our modeling gives very accurate results and has a well volatility forecasting performance.

Keywords: Multivariate periodic stochastic volatility; periodic stationarity; periodic Kalman filter; particle filtering; exchange rates; Saharan Blend oil.

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1. Introduction

Instantaneous volatility and volatility clustering modeling play an important role in the analysis of financial time series. Since the introduction of the Autoregressive Conditional Heteroskedasticity ($ARCH$) model in the seminal paper of Engle (1982), a significant part of the literature was devoted to these issues. However and despite its success, this modeling exhibits some drawbacks. A satisfactory alternative to $ARCH$-type models was introduced by Taylor (1982). The namely stochastic volatility (SV) model allows the variance of the returns to be an unobserved random process. This is not the case in $ARCH$ and their generalized $GARCH$ models, where volatility is a function of previous observations and/or past volatility. Moreover, the SV model allows the log of the volatility to evolve. This ensures that the variance of the process always remains positive without the need for further constraints, as is the case in the $ARCH/GARCH$ models.

Apart from the volatility clustering phenomenon, there are other important stylized facts associated with financial returns series. One can cite excess kurtosis, asymmetry and heavy-tailed errors, persistence (long memory property), etc. In practice, it turns out that a large part of the literature was based on univariate models. However, some stylized facts cannot be captured by a univariate description. This is, for example, the case of the covariation effect, i.e. the study of the relationships between the volatilities and covolatilities of several markets. On one hand, much of the financial decision making (such as portfolio optimization, asset allocation, risk management, and asset pricing) clearly needs to take correlations into account. On the other hand, it is now well documented that financial volatilities of different assets and markets move together over time. Large changes in one asset are matched by large movements in another.

This fact plays a critical role in the construction of the appropriate modeling of financial time series. The multivariate models for modeling time-varying volatility are particularly interesting. Indeed, they can lead to greater statistical efficiency. As a result, working with multivariate modeling framework leads to more relevant empirical models than with separate univariate models. Some multivariate stochastic volatility models have even recently become a major concern in the investigation of the correlation structure of multivariate economic series in general, and of multivariate financial time series in particular. Various extensions of the basic Multivariate SV (MSV) models have been proposed in the literature. One can cite for example Harvey et al. (1994), Daníelsson (1998), Jungbacker and Koopman (2006), Smith and Pitts (2006), Chan et al. (2006), Asai and McAleer ($2009a, b$), etc. Asai et al. (2006) propose a detailed review of the literature. Note that these models have attracted a lot of attention in modern finance theory and enjoyed voluminous empirical applications. Yu and Meyer (2006) thoroughly discuss a wide range of dominant MSV models, available in the literature, on specification, estimation, and evaluation of MSV models. It is worth noting that all of the proposed models deal with constant volatility parameters. There seems to be no formulation that can adequately explain multivariate time series whose structure changes over time. This is, in particular, the case for time series with a volatility displaying a periodic correlation pattern. For instance, it is usually observed in financial time series for which correlations between daily returns and volatility display some day-of-the-week effects (see e.g. Franses and Paap (2000), Bentarzi and Hamdi (2008) and Hamdi and Souam (2017)). Hence, it is interesting to assume that the log-volatilities in each day (more generally, season) might be described by a different model. More specifically, we can assume that the parameters change periodically over time.

The contribution of our paper is twofold. We, firstly, propose a model, called Periodic Multivariate Autoregressive Stochastic Volatility ($PVAR$-SV) model, that allows the Granger causality in volatility in order to capture periodicity in stochastic conditional variance. We, secondly, provide an empirical application which shows that our modeling gives very accurate results and has a well volatility forecasting performance.

The remainder of the paper is organized as follows. Section 2 thoroughly describes the $PVAR$-SV models. In Section 3, we provide some probabilistic properties of this class of models. In Section 4, we propose two straightforward methods for the estimation problem, for which the implementation is based on periodic Kalman filter and on the particle filter and smoother with expectation-maximization (EM) algorithm. Finally, in Section 5 we apply our model to a set of three exchange rates and oil price time series, namely U.S. dollar/Algerian dinar (USD/DZD), Euro/Algerian dinar (EUR/DZD), Euro/U.S. dollar (EUR/USD) and the daily Saharan Blend oil prices.

2. Multivariate Periodic Autoregressive Stochastic Volatility

The univariate periodic autoregressive stochastic volatility model (PAR-SV) was introduced by Aknouche et al. (2007) and thoroughly studied by Boussaha and Hamdi (2015). It provides a successful alternative to the class of periodic generalized autoregressive conditionally heteroscedastic ($PGARCH$) models (Bollerslev and Ghysels, 1996). Indeed, it takes into account the time-varying and persistent volatility as well as the periodicity feature in the autocorrelation structure exhibited by many nonlinear time series.

A stochastic process $\{\varepsilon_t; t \in \mathbb{Z}\}$ has a PAR-SV representation of period S, if it is a solution of the following stochastic difference equation

$$\left\{ \begin{array}{l} \varepsilon_t = \eta_t \exp\left\{\frac{1}{2}x_t\right\}, \\ x_t = \alpha_t + \beta_t x_{t-1} + Q_t e_t, \end{array} \right. \quad t \in \mathbb{Z}, \tag{1}$$

where the parameters α_t, β_t and Q_t are periodic in t with period S (i.e., $\alpha_{t+\tau S} = \alpha_t$, $\beta_{t+\tau S} = \beta_t$ and $Q_{t+\tau S} = Q_t$, for all $t, \tau \in \mathbb{Z}$). The two independent sequences (η_t) and (e_t) are independent and identically distributed ($i.i.d.$) random variables with zero mean and unit variance.

In the current paper, we propose a multivariate periodic autoregressive stochastic volatility ($PVAR$-SV) model. Let $\varepsilon_t = (\varepsilon_{t,1}, ..., \varepsilon_{t,m})'$ be the $m \times 1$ vector of stock returns at time t and $X_t = (X_{t,1}, ..., X_{t,m})'$ be the corresponding vector of log-variances. The $PVAR$-SV model can be defined as follows

$$\left\{ \begin{array}{l} \varepsilon_t = V_t^{1/2} \eta_t, \\ X_t = \alpha_t + \Phi_t X_{t-1} + \mathbf{e}_t, \end{array} \right. \quad t \in \mathbb{Z}, \tag{2}$$

where $V_t = diag\left(\exp X_{t,1}, ..., \exp X_{t,m}\right) = diag\left(\exp X_t\right)$, $\Phi_t = \left(\phi_{i,j}^{(t)}\right)_{i,j=1,...,m}$ is a lower triangular matrix, i.e., $\phi_{i,j}^{(t)} = 0$ if $i < j$, and $\alpha_t = \left(\alpha_{t,1}, \alpha_{t,2}, .., \alpha_{t,m}\right)'$. The two vectors $\eta_t = (\eta_{t,1}, ..., \eta_{t,m})'$ and $\mathbf{e}_t = (\mathbf{e}_{t,1}, ..., \mathbf{e}_{t,m})'$ represent zero-mean $i.i.d.$ random processes with

$$\mathbb{E}\left(\eta_t \eta_t'\right) = \Sigma_\eta^{(t)}, \ \mathbb{E}\left(\mathbf{e}_t \mathbf{e}_t'\right) = \Sigma_{\mathbf{e}}^{(t)} \text{ and } \mathbb{E}\left(\eta_t \mathbf{e}_t'\right) = \mathbf{0}_m,$$

where $\Sigma_\eta^{(t)}$ and $\Sigma_{\mathbf{e}}^{(t)}$ are $m \times m$ nonnegative definite matrices. The parameters α_t, Φ_t, and the matrices $\Sigma_\eta^{(t)}$, $\Sigma_{\mathbf{e}}^{(t)}$ are periodic in t, with period S.

To emphasize the periodicity, let $t = s + \tau S$, for $\tau \in \mathbb{Z}$ and $1 \leq s \leq S$. Then, model (2) can be written in the following form

$$\left\{ \begin{array}{l} \varepsilon_{s+\tau S} = diag\left(\exp\left\{\frac{1}{2}X_{s+\tau S}\right\}\right)\eta_{s+\tau S}, \\ X_{s+\tau S} = \alpha_s + \Phi_s X_{s+\tau S-1} + \mathbf{e}_{s+\tau S}, \end{array} \right. \quad \tau \in \mathbb{Z}, \ 1 \leq s \leq S. \tag{3}$$

3. Periodic stationarity and computation of moments

Let us now investigate some of the basic properties of the $PVAR$-SV model (2). For the development of statistical estimation and testing theory, we need to provide conditions for periodic strict stationarity and periodic ergodicity (in the sense given, for example, by Aknouche and Bibi (2009)) of a multivariate *PAR-SV* process. It is clear from the multiplicative form of the second equation in model (2) that the process $\{X_t; t \in \mathbb{Z}\}$ is strictly periodically stationary and periodically ergodic if and only if the spectral radius of the matrix $\prod_{v=1}^{S} \Phi_v$ is less than one.

In order to go further, let us introduce some notation. Let $A^{\otimes r} = A \otimes A \otimes \ldots \otimes A$ be the r-th Kronecker power of any matrix A, where $r \in \{0, 1, ...\}$ (by convention $A^{\otimes 0} = \mathbf{I}$ and $A^{\otimes 1} = A$). Denote by $\rho(A)$ the spectral radius of any square matrix A.

It may be noted that if $\rho\left(\prod_{v=1}^{S} \Phi_v\right) < 1$, then $\{X_t;\ t \in \mathbb{Z}\}$ is a periodic stationary process with the unconditional periodic mean, for $\tau \in \mathbb{Z}$ and $s \in \{1, 2, ..., S\}$,

$$\mu_X^{(s)} := \mathbb{E}\left(X_{s+\tau S}\right) = \left(\mathbf{I}_m - \prod_{v=0}^{S-1} \Phi_{s-v}\right)^{-1} \sum_{v=0}^{S-1} \left(\prod_{i=0}^{v-1} \Phi_{s-i}\right) \alpha_{s-v},$$

and the unconditional periodic second order moment

$$\begin{aligned} vec\left(\Sigma_X^{(s)}\right) &:= \mathbb{E}\left(X_{s+\tau S}^{\otimes 2}\right) \\ &= \left(\mathbf{I}_{m^2} - \left(\prod_{v=0}^{S-1} \Phi_{s-v}\right)^{\otimes 2}\right)^{-1} \sum_{v=0}^{S-1} \left(\prod_{i=0}^{v-1} \Phi_{s-i}\right)^{\otimes 2} vec\left(\Sigma_{\mathbf{e}}^{(s-v)}\right), \end{aligned}$$

where $vec(A)$ denotes an $n_1 n_2$-vector, obtained from an $(n_1 \times n_2)$-matrix A by stacking its columns in the natural order.

If $\rho\left(\prod_{v=1}^{S} \Phi_v\right) < 1$, the process X_t is given as a first order periodic VAR causal model which can be represented as

$$X_{s+\tau S} = \mu_X^{(s)} + \sum_{l \geq 0} \left(\prod_{i=0}^{l-1} \Phi_{s-i}\right) \mathbf{e}_{s+\tau S-l}.$$

Since ε_t is the product of two strictly periodically stationary processes, it must also be strictly periodically stationary. Thus the conditions needed to ensure the periodic stationarity of ε_t (in the strict and weak senses), are just the ones needed to ensure periodic stationarity of the process X_t. This is summarized in the following theorem.

Theorem 1. *Model* (2) *admits a unique nonanticipative (future-independent) strictly periodically stationary solution given, for* $\tau \in \mathbb{Z}$ *and* $s \in \{1, 2, ..., S\}$, *by*

$$\varepsilon_{s+\tau S} = diag\left(\exp\left\{\frac{1}{2}\left(\mu_X^{(s)} + \sum_{l\geq 0}\left(\prod_{i=0}^{l-1}\Phi_{s-i}\right)\mathbf{e}_{s+\tau S-l}\right)\right\}\right)\eta_{s+\tau S}, \qquad (4)$$

where the series in (4) *converges almost surely, if and only if*

$$\rho\left(\prod_{v=1}^{S}\Phi_v\right) < 1. \qquad (5)$$

In the sequel, we explicitly characterize the variance, skewness and kurtosis of the *PVAR-SV* model (2). Indeed, it turns out that these moments are very useful to describe data. This is why they are widely used by most researchers.

Suppose that (5) holds and the two vectors η_t and e_t are multivariate normal. In a first step, we provide very general results. In a second step, we do consider normality in order to get more explicit formula for these moments. We thus have:

$$\begin{aligned}
\mathbb{E}\left(\varepsilon_t^{\otimes r}\right) &= \mathbb{E}\left(\left(V_t^{1/2}\eta_t\right)^{\otimes r}\right) \\
&= \mathbb{E}\left[diag\left\{\left(\exp\frac{1}{2}X_t\right)^{\otimes r}\right\} \times \eta_t^{\otimes r}\right] \\
&= diag\left\{\mathbb{E}\left[\left(\exp\frac{1}{2}X_t\right)^{\otimes r}\right]\right\}\mathbb{E}\left[\eta_t^{\otimes r}\right] \\
&= diag\left\{\mathbb{E}\left[Y_t^{(r)}\right]\right\}\mathbb{E}\left[\eta_t^{\otimes r}\right],
\end{aligned}$$

where $Y_t^{(r)} = \left(\exp\frac{1}{2}X_t\right)^{\otimes r} = \left(Y_{t,1}^{(r)}, Y_{t,2}^{(r)}, ..., Y_{t,m^r}^{(r)}\right)$, with

$$Y_{t,i}^{(r)} = \exp\left(\frac{1}{2}\sum_{n=0}^{r-1}X_{t,k_{i,n}}\right), \text{ for } i = 1, ..., m^r,$$

with

$$i - k_{i,0} = (k_{i,1} - 1)\,m + \cdots + (k_{i,r-1} - 1)\,m^{r-1},$$

and

$$k_{i,0} = \begin{cases} m & \text{if } i \equiv 0 \pmod{m}, \\ i \equiv 0 \pmod{m} & \text{otherwise.} \end{cases}$$

From this last equation, it turns out that

$$\mathbb{E}\left(Y_{t,i}^{(r)}\right) = \mathbb{E}\left(\exp\left\{T_{i,r}' X_t\right\}\right) = M_{X_t}\left(T_{i,r}\right),\ i = 1, ..., m^r$$

where $M_{X_t}(T) = \mathbb{E}\left(\exp\left\{T' X_t\right\}\right)$ denotes the moment generating function of X_t, $T_{i,r} = \frac{1}{2}\sum_{n=0}^{r-1} w_{k_{i,n}}$, and w_j is the $m \times 1$ canonical vector $w_j = \left(\mathbf{0}_{1\times(j-1)}, 1, \mathbf{0}_{1\times m-j}\right)'$. Thus,

$$\begin{aligned} \mathbb{E}\left(\varepsilon_t^{\otimes r}\right) &= diag\left\{\left(M_{X_t}\left(T_{1,r}\right), ..., M_{X_t}\left(T_{m^r,r}\right)\right)\right\} \mathbb{E}\left[\eta_t^{\otimes r}\right] \\ &= diag\left\{\left(T_{1,r}'\mu_X^{(t)} + \frac{1}{2}T_{1,r}'\Sigma_X^{(t)}T_{1,r}, ..., T_{m^r,r}'\mu_X^{(t)} + \frac{1}{2}T_{m^r,r}'\Sigma_X^{(t)}T_{m^r,r}\right)\right\} \\ &\times \mathbb{E}\left[\eta_t^{\otimes r}\right]. \end{aligned}$$

Given the matrix expressions of the third and fourth ($r = 3$ and 4) moments of the *PVAR-SV* process $\{\varepsilon_t; t \in \mathbb{Z}\}$, we can derive the exact matrix expression formulae of the skewness and kurtosis measures. These expressions are only functions of the parameters of the model. Let us note that non-linearities are typically analyzed through these pairwise measures in finance. They are defined as follows:

$$sk_\varepsilon^{(t)} := \mathbb{E}\left(\left[\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{-1/2}\varepsilon_t\right]^{\otimes 3}\right)' \mathbb{E}\left(\left[\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{-1/2}\varepsilon_t\right]^{\otimes 3}\right),$$

and

$$\kappa_\varepsilon^{(t)} := vec\left(\mathbf{I}_{m^2}\right)' \mathbb{E}\left(\left[\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{-1/2}\varepsilon_t\right]^{\otimes 4}\right),$$

where $\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{1/2}$ is any symmetric square root of the variance-covariance matrix of ε_t (see e.g. Kollo, 2008).

In the *PVAR-SV* case, we have

$$\begin{aligned} \mathbb{E}\left(\left[\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{-1/2}\varepsilon_t\right]^{\otimes 3}\right) &= \left(\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{-1/2}\right)^{\otimes 3} \\ &\times diag\left\{\left(T_{1,3}'\mu_X^{(t)} + \frac{1}{2}T_{1,3}'\Sigma_X^{(t)}T_{1,3}, ..., T_{m^3,3}'\mu_X^{(t)} + \frac{1}{2}T_{m^3,3}'\Sigma_X^{(t)}T_{m^3,3}\right)\right\} \mathbb{E}\left[\eta_t^{\otimes 3}\right] \\ &= \mathbf{0}_{m^3\times 1}, \end{aligned}$$

since $\mathbb{E}\left[\eta_t^{\otimes 3}\right] = \mathbf{0}_{m^3\times 1}$, and

$$\mathbb{E}\left(\left[\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{-1/2}\varepsilon_t\right]^{\otimes 4}\right) = \left[\left(\Sigma_{\varepsilon,0}^{(t)}\right)^{-1/2}\right]^{\otimes 4}$$

$$\times\, diag\left\{\left(T_{1,4}'\mu_X^{(t)} + \frac{1}{2}T_{1,4}'\Sigma_X^{(t)}T_{1,4}, ..., T_{m^4,4}'\mu_X^{(t)} + \frac{1}{2}T_{m^4,4}'\Sigma_X^{(t)}T_{m^4,4}\right)\right\}\mathbb{E}\left[\eta_t^{\otimes 4}\right].$$

Note that under the normality hypothesis, the closed form of $\mathbb{E}\left[\eta_t^{\otimes r}\right]$ can be obtained from the result provided by Kollo and von Rosen (2006, Corollary 2.2.7.4). Indeed, odd moments of η_t are equal to zero and even moments are given by the following equalities:

$$\mathbb{E}\left(\eta_t^{\otimes 2}\right) = vec\left(\Sigma_\eta^{(t)}\right),$$

$$\mathbb{E}\left(\eta_t^{\otimes 4}\right) = \left(\mathbf{I}_{m^4} + \mathbf{I}_m \otimes K_{m,m} \otimes \mathbf{I}_m + \mathbf{I}_m \otimes K_{m^2,m}\right)\left[vec\left(\Sigma_\eta^{(t)}\right)\right]^{\otimes 2},$$

and

$$\mathbb{E}\left(\eta_t^{\otimes r}\right) = \sum_{i=2}^{r}\left(\mathbf{I}_m \otimes K_{m^{i-2},m} \otimes \mathbf{I}_{m^{r-i}}\right)\left[vec\left(\Sigma_\eta^{(t)}\right) \otimes \mathbb{E}\left(\eta_t^{\otimes r-2}\right)\right], \text{ for } r = 2, 4, 6, ...$$

where $K_{p,q}$ is the $pq \times pq$ commutation matrix. The following proposition summarizes the exact general formula of the different moments and gives the variance, the skewness and the kurtosis of the $PVAR$-SV model.

Proposition 2. *For a periodic stationary solution $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ of the $PVAR$-SV model defined by (2), for any positive integer r and under the normality hypothesis of the two vectors η_t and e_t,we have*

$$\begin{aligned}\mu_{\varepsilon^r}^{(s)} &:= \mathbb{E}\left(\varepsilon_{s+\tau S}^{\otimes r}\right)\\ &= diag\left\{\left(T_{1,r}'\mu_X^{(s)} + \frac{1}{2}T_{1,r}'\Sigma_X^{(s)}T_{1,r}, ..., T_{m^r,r}'\mu_X^{(s)} + \frac{1}{2}T_{m^r,r}'\Sigma_X^{(s)}T_{m^r,r}\right)\right\}\\ &\times \mathbb{E}\left[\eta_{s+\tau S}^{\otimes r}\right].\end{aligned}$$

Furthermore, the variance, skewness and the kurtosis of the distribution of $\varepsilon_{s+\tau S}$ are given by:

$$\begin{aligned}vec\left(\Sigma_{\varepsilon,0}^{(s)}\right) &:= \mathbb{E}\left(\varepsilon_{s+\tau S}^{\otimes 2}\right)\\ &= diag\left\{\left(T_{1,2}'\mu_X^{(s)} + \frac{1}{2}T_{1,2}'\Sigma_X^{(s)}T_{1,2}, ..., T_{m^2,2}'\mu_X^{(s)} + \frac{1}{2}T_{m^2,2}'\Sigma_X^{(s)}T_{m^2,2}\right)\right\}\\ &\times vec\left(\Sigma_\eta^{(s)}\right),\\ sk_\varepsilon^{(s)} &= 0,\end{aligned}$$

and

$$\kappa_{\varepsilon}^{(s)} := vec\left(\mathbf{I}_{m^2}\right)'\left[\left(\Sigma_{\varepsilon,0}^{(s)}\right)^{-1/2}\right]^{\otimes 4}$$
$$\times\, diag\left\{\left(T_{1,4}'\mu_X^{(s)}+\frac{1}{2}T_{1,4}'\Sigma_X^{(s)}T_{1,4},...,T_{m^4,4}'\mu_X^{(s)}+\frac{1}{2}T_{m^4,4}'\Sigma_X^{(s)}T_{m^4,4}\right)\right\}$$
$$\times\left(\mathbf{I}_{m^4}+\mathbf{I}_m\otimes K_{m,m}\otimes\mathbf{I}_m+\mathbf{I}_m\otimes K_{m^2,m}\right)\left[vec\left(\Sigma_{\eta}^{(s)}\right)\right]^{\otimes 2}.$$

4. The autocorrelation functions of the *PVAR-SV* model

In this section, we investigate the dependence structure in a periodic stationary solution $\{\varepsilon_t; t\in\mathbb{Z}\}$ of (2). We, more particularly, examine its autocovariance function, most frequently used by statisticians, time series analysts and practitioners.

Let us first recall that the periodic autocovariance function of a centered m-variate periodically stationary process $\{\varepsilon_t; t\in\mathbb{Z}\}$ is defined by

$$\Sigma_{\varepsilon,h}^{(s)} := cov\left(\varepsilon_{s+\tau S},\varepsilon_{s+\tau S-h}\right)=\mathbb{E}\left(\varepsilon_t\varepsilon_{t-h}'\right),\ h=0,1,2,...$$

It is more convenient, in practice, to work with the vector autocovariance function defined by

$$vec\left(\Sigma_{\varepsilon,h}^{(s)}\right) := \mathbb{E}\left(\varepsilon_t\otimes\varepsilon_{t-h}\right).$$

The following proposition completely characterizes this vector autocovariance function.

Proposition 3. *Let $\{\varepsilon_t; t\in\mathbb{Z}\}$ be a periodic stationary solution of* (2)*. Then $\{\varepsilon_t;\ t\in\mathbb{Z}\}$ is a periodic weak white noise process with variance given, for $s=1,...,S,$ by*

$$vec\left(\Sigma_{\varepsilon,0}^{(s)}\right)=diag\left\{\left(T_{1,2}'\mu_X^{(s)}+\frac{1}{2}T_{1,2}'\Sigma_X^{(s)}T_{1,2},...,T_{m^2,2}'\mu_X^{(s)}+\frac{1}{2}T_{m^2,2}'\Sigma_X^{(s)}T_{m^2,2}\right)\right\}$$
$$\times\, vec\left(\Sigma_{\eta}^{(s)}\right).$$

PROOF. For $h > 0$, we have

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_t \otimes \varepsilon_{t-h}\right] &= \mathbb{E}\left[\left(diag\left(\exp\left\{\frac{X_t}{2}\right\}\right)\eta_t\right) \otimes \left(diag\left(\exp\left\{\frac{X_{t-h}}{2}\right\}\right)\eta_{t-h}\right)\right] \\
&= \mathbb{E}\left[diag\left\{\left(\exp\left\{\frac{X_t}{2}\right\}\right) \otimes \left(\exp\left\{\frac{X_{t-h}}{2}\right\}\right)\right\}\left(\eta_t \otimes \eta_{t-h}\right)\right] \\
&= diag\left(\mathbb{E}\left[\exp\left\{\frac{X_t}{2}\right\} \otimes \exp\left\{\frac{X_{t-h}}{2}\right\}\right]\right) \times \mathbb{E}\left(\eta_t \otimes \eta_{t-h}\right) \\
&= diag\left(\mathbb{E}\left[Y_t^{(1)} \otimes Y_{t-h}^{(1)}\right]\right) \times \mathbb{E}\left(\eta_t\right) \otimes \mathbb{E}\left(\eta_{t-h}\right) \\
&= diag\left(\mathbb{E}\left[\exp\left\{\tfrac{X_{t,1}+X_{t-h,1}}{2}, ..., \tfrac{X_{t,1}+X_{t-h,m}}{2}, ..., \tfrac{X_{t,m}+X_{t-h,1}}{2},\right.\right.\right. \\
&\quad \left.\left.\left. ..., \tfrac{X_{t,m}+X_{t-h,m}}{2}\right\}\right]\right) \times \mathbb{E}\left(\eta_t\right) \otimes \mathbb{E}\left(\eta_{t-h}\right) \\
&= \mathbf{0}_{m^2 \times 1},
\end{aligned}$$

since (η_t) is a centered sequence and $X_{t,i}$ and $X_{t-h,j}$ are two Gaussian random variables for all $i, j = 1, ..., m$, then $\mathbb{E}\left[\exp\left\{\frac{X_{t,i}+X_{t-h,j}}{2}\right\}\right] < \infty$, $\forall i, j = 1, ..., m$. This defines the multivariate periodic *AR-SV* framework, in which there is no linear dependence structure in $\{\varepsilon_t;\ t \in \mathbb{Z}\}$. Therefore, even though $\{\varepsilon_t; t \in \mathbb{Z}\}$ is a periodic white noise process, it is still possible that non-linear forms of dependence between the successive terms of $\{\varepsilon_t; t \in \mathbb{Z}\}$ exist. The nature of the dependence structure in the periodic *VAR-SV* process defined in (2) can be obtained by studying the covariance structure of the observations raised to the power $r \in \mathbb{N}^*$. We have from (2)

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_t^{\otimes 2r} \otimes \varepsilon_{t-h}^{\otimes 2r}\right] &= \mathbb{E}\left[\left(diag\left(\exp\left\{\tfrac{X_t}{2}\right\}\right)\eta_t\right)^{\otimes 2r} \otimes \left(diag\left(\exp\left\{\tfrac{X_{t-h}}{2}\right\}\right)\eta_{t-h}\right)^{\otimes 2r}\right] \\
&= diag\left(\mathbb{E}\left[\left(\exp\left\{\tfrac{X_t}{2}\right\}\right)^{\otimes 2r} \otimes \left(\exp\left\{\tfrac{X_{t-h}}{2}\right\}\right)^{\otimes 2r}\right]\right) \\
&\quad \times \mathbb{E}\left(\eta_t^{\otimes 2r}\right) \otimes \mathbb{E}\left(\eta_{t-h}^{\otimes 2r}\right) \\
&= diag\left(\mathbb{E}\left[Y_t^{(2r)} \otimes Y_{t-h}^{(2r)}\right]\right)\mathbb{E}\left(\eta_t^{\otimes 2r} \otimes \eta_{t-h}^{\otimes 2r}\right) \neq \mathbf{0}_{m^{4r} \times 1},
\end{aligned}$$

since the first elements of the two vectors $Y_t^{(2r)} \otimes Y_{t-h}^{(2r)}$ and $\eta_t^{\otimes 2r} \otimes \eta_{t-h}^{\otimes 2r}$ are given by

$$\left(Y_t^{(2r)} \otimes Y_{t-h}^{(2r)}\right)_1 = \exp\left\{r\left(X_{t,1} + X_{t-h,1}\right)\right\} \text{ and } \left(\eta_t^{\otimes 2r} \otimes \eta_{t-h}^{\otimes 2r}\right)_1 = \eta_{t,1}^{2r}\eta_{t-h,1}^{2r},$$

from which we can easily see that the first element of vector $\mathbb{E}\left[\varepsilon_t^{\otimes 2r} \otimes \varepsilon_{t-h}^{\otimes 2r}\right]$

$$\begin{aligned}\left(\mathbb{E}\left[\varepsilon_t^{\otimes 2r} \otimes \varepsilon_{t-h}^{\otimes 2r}\right]\right)_1 &= \left(\mathbb{E}\left[Y_t^{(2r)} \otimes Y_{t-h}^{(2r)}\right]\right)_1 \left(\mathbb{E}\left[\eta_t^{\otimes 2r} \otimes \eta_{t-h}^{\otimes 2r}\right]\right)_1 \\ &= \mathbb{E}\left(\exp\left\{r\left(X_{t,1} + X_{t-h,1}\right)\right\}\right) \mathbb{E}\left(\eta_{t,1}^{2r}\right) \mathbb{E}\left(\eta_{t-h,1}^{2r}\right) \\ &= \mathbb{E}\left(\exp\left\{r\left(X_{t,1} + X_{t-h,1}\right)\right\}\right) \left(\frac{(2r)!}{2^r r!}\right)^2 \left(\sigma_{\eta_1}^{(t)} \sigma_{\eta_1}^{(t-h)}\right)^{2r} > 0,\end{aligned}$$

where $\sigma_{\eta_1}^{(t)}$ is the standard deviation of the first component of the random vector η_t.

5. Estimation methodology

In this section, we propose two methodologies in order to estimate our model. The first one is the quasi-maximum likelihood method based on periodic Kalman filter. The second one uses the maximum likelihood and is based on the *EM* algorithm with particle filters and smoothers. We, more particularly, thoroughly describe the second one which turns out to be more useful in practice.

5.1. Quasi-maximum likelihood method based on periodic Kalman filter

In order to estimate the parameters of the multivariate non-periodic *AR-SV* model, Harvey et al. (1994) proposed a quasi-maximum likelihood (*QML*) method. It is based on the state-space representation after setting the observed variable as a logarithm of the vector $\varepsilon_t \odot \varepsilon_t$, where the operator $\odot$ is the Hadamard (or element-by-element) product. In our periodic case, model (3) can be linearized by taking the same transformation $\ln\left(\varepsilon_{s+\tau S} \odot \varepsilon_{s+\tau S}\right)$ where we get the following state-space representation

$$\left\{\begin{array}{l} Z_{s+\tau S} = X_{s+\tau S} + d_s + \mathbf{u}_{s+\tau S}, \\ X_{s+\tau S} = \alpha_s + \Phi_s X_{s+\tau S-1} + \mathbf{e}_{s+\tau S}, \end{array}\right. \quad \tau \in \mathbb{Z},\ 1 \leq s \leq S, \tag{6}$$

where $Z_{s+\tau S} = \left(Z_{s+\tau S,1}, Z_{s+\tau S,2}, ..., Z_{s+\tau S,m}\right)'$, $Z_{s+\tau S} = \ln\left(\varepsilon_{s+\tau S} \odot \varepsilon_{s+\tau S}\right)$, $d_s = \mathbb{E}\left[\ln\left(\eta_{s+\tau S} \odot \eta_{s+\tau S}\right)\right]$ and

$$\mathbf{u}_{s+\tau S} = \ln\left(\eta_{s+\tau S} \odot \eta_{s+\tau S}\right) - \mathbb{E}\left[\ln\left(\eta_{s+\tau S} \odot \eta_{s+\tau S}\right)\right].$$

Let us remark that even though the random process $\eta_{s+\tau S}$ is Gaussian, the measurement equation errors, $\mathbf{u}_{s+\tau S}$, in (6) are nonnormal. The mean vector

d_s of $\ln\left(\eta_{s+\tau S} \odot \eta_{s+\tau S}\right)$ and the (i,j)-th element of the covariance matrix $\Sigma_{\mathbf{u}}^{(s)}$ of $\mathbf{u}_{s+\tau S}$ can be given (see Harvey et al, 1994) respectively for $s = 1, ..., S$, by

$$d_s = \left(-1.2749 + \log\left(\Sigma_\eta^{(s)}(1,1)\right), \ldots, -1.2749 + \log\left(\Sigma_\eta^{(s)}(m,m)\right)\right)',$$

and

$$\Sigma_{\mathbf{u}}^{(s)}(i,j) = \begin{cases} \sum\limits_{n=1}^{\infty} \frac{(n-1)!}{n\left(\prod_{k=1}^{n}\left(\frac{1}{2}+k-1\right)\right)} \left(\frac{\Sigma_\eta^{(s)}(i,j)}{\sqrt{\Sigma_\eta^{(s)}(i,i)}\sqrt{\Sigma_\eta^{(s)}(j,j)}}\right)^{2n}, & \text{for } i \neq j,\ i,j = 1, ..., m, \\ \frac{\pi^2}{2}, & \text{otherwise.} \end{cases}$$

This last representation of the model makes the estimation problem more evident via QML method as suggested by Harvey et al. (1994) for the multivariate non periodic case and Boussaha and Hamdi (2015) for the univariate periodic case. However, the transformation of the model causes loss of information in the multivariate case. More specifically, an estimation could be made about the absolute values of the unknown parameters in $\Sigma_\eta^{(s)}$, $s = 1, ..., S$, namely the $\Sigma_\eta^{(s)}(i,j)$'s, and the covariances between different $\eta_{t,i}$'s, but their signs could not be estimated. This is due to the loss of information when the observations are squared. To solve this issue, Harvey et al. (1994) suggested to use the signs of the untransformed observations to obtain the sign of the covariance coefficients. They have proposed to estimate the sign of $\Sigma_\eta^{(s)}(i,j)$ as positive if more than one-half of the pairs $\varepsilon_{t,i}\varepsilon_{t,j}$ are positive.

To discuss the QML method based on periodic Kalman filter, let $\widehat{Z}_{s+\tau S|s+\tau S-1}$ be the best linear (one step ahead) predictor of $Z_{s+\tau S}$ based on $Z_1, Z_2, ...,$ $Z_{s+\tau S-1}$, and let $\widehat{\mathbf{u}}_{s+\tau S} = Z_{s+\tau S} - \widehat{Z}_{s+\tau S|s+\tau S-1}$ be the sample innovation at time $s + \tau S$, with mean square error $\Omega_{s+\tau S} = \mathbb{E}\left(\widehat{\mathbf{u}}_{s+\tau S}\widehat{\mathbf{u}}'_{s+\tau S}\right)$.

The innovation at time $s + \tau S$ is defined as

$$\widehat{\mathbf{u}}_{s+\tau S} = Z_{s+\tau S} - \widehat{X}_{s+\tau S|s+\tau S-1} - d_s,$$

where $\widehat{X}_{s+\tau S|s+\tau S-1}$ is the best linear predictor of $X_{s+\tau S}$ based on $Z_1, Z_2, ...,$ $Z_{s+\tau S-1}$, with mean square error

$$P_{s+\tau S|s+\tau S-1} = \mathbb{E}\left[\left(X_{s+\tau S} - \widehat{X}_{s+\tau S|s+\tau S-1}\right)\left(X_{s+\tau S} - \widehat{X}_{s+\tau S|s+\tau S-1}\right)'\right].$$

For a given realization $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ generated from model (2), the quasi-likelihood of the parameter vector $\theta = (\theta'_1, ..., \theta'_S)'$, where $\theta_s =$

$\left(\alpha_s, (vech\,(\Phi_s))', \left(vech\left(\Sigma_{\mathbf{e}}^{(s)}\right)\right)', \left(vech\left(\Sigma_{\eta}^{(s)}\right)\right)'\right)'$, can be written in the innovation form as follows

$$L(\theta; Z) = (2\pi)^{-\frac{nm}{2}} \prod_{s=1}^{S} \prod_{\tau=0}^{\tau_1 - 1} (\det \Omega_{s+\tau S})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\widehat{\mathbf{u}}'_{s+\tau S}\Omega^{-1}_{s+\tau S}\widehat{\mathbf{u}}_{s+\tau S}\right\}, \quad (7)$$

in which we need to evaluate $\Omega_{s+\tau S}$ and $\widehat{\mathbf{u}}_{s+\tau S}$, for $s = 1, ..., S$ and $\tau = 0, ..., \tau_1 - 1$, where, for simplicity purposes, n can be taken as $n = \tau_1 S$. Note that the sample innovation and its mean square error involved in (7) can be recursively computed using either the Kalman filter (Kalman, 1960) or the periodic Chandrasekhar-type recursions (Aknouche and Hamdi, 2007). Clearly, from the state-space representation (6), the periodic Kalman filter, as it is well known, is given by the following recursions:

$$\begin{cases} \widehat{\mathbf{u}}_{s+\tau S} = Z_{s+\tau S} - \widehat{X}_{s+\tau S|s+\tau S-1} - d_s, \\ \Omega_{s+\tau S} = P_{s+\tau S|s+\tau S-1} + \Sigma_{\mathbf{u}}^{(s)}, \\ K_{s+\tau S} = \Phi_{s+1} P_{s+\tau S|s+\tau S-1} \Omega^{-1}_{s+\tau S}, \\ \widehat{X}_{s+\tau S+1|s+\tau S} = \alpha_{s+1} + \Phi_{s+1}\widehat{X}_{s+\tau S|s+\tau S-1} + K_{s+\tau S}\widehat{\mathbf{u}}_{s+\tau S}, \\ P_{s+\tau S+1|s+\tau S} = \Sigma_{\mathbf{e}}^{(s+1)} \\ \qquad +\Phi_{s+1}\left(P_{s+\tau S|s+\tau S-1} - P_{s+\tau S|s+\tau S-1}\Omega^{-1}_{s+\tau S}P_{s+\tau S|s+\tau S-1}\right)\Phi'_{s+1}, \end{cases}$$

with start-up values

$$\widehat{X}_{1|0} = \left(\mathbf{I}_m - \prod_{v=0}^{S-1} \Phi_{1-v}\right)^{-1} \sum_{v=0}^{S-1} \left(\prod_{i=0}^{v-1} \Phi_{1-i}\right) \alpha_{1-v},$$

and

$$vec\left(P_{1|0}\right) = \left(\mathbf{I}_{m^2} - \left(\prod_{v=0}^{S-1} \Phi_{1-v}\right)^{\otimes 2}\right)^{-1} \times \sum_{v=0}^{S-1} \left(\prod_{i=0}^{v-1} \Phi_{1-i}\right)^{\otimes 2} \left[\left(\alpha_{1-v} - \mu_X^{(1-v)} + \Phi_{1-v}\mu_X^{(S-v)}\right)^{\otimes 2} + vec\Sigma_{\mathbf{e}}^{(1-v)}\right].$$

In practice, it is very difficult to obtain explicit formula of the QML estimator, $\widehat{\theta}$, which maximizes (7), with respect to θ. One thus needs to use numerical methods. It is also important to note that, under appropriate conditions, this QML estimator, $\widehat{\theta}$, has been shown to be consistent and asymptotically normally distributed (Ljung and Caines, 1979, p. 36). Despite these desirable asymptotic properties, the approximations provided by the periodic Kalman filter become less effective in our case where the normality hypothesis is abandoned. Indeed, the QML estimator is not necessarily the best estimator for finite samples. For the periodic linear state-space models and when the sample size is small, Guerbyenne and Hamdi (2015) have provided numerous examples which indicate the poor finite sample properties of $\widehat{\theta}$. To improve the finite sample performance of the QML estimator, they suggested to use some procedures based on a bootstrap method to fit a periodic time series model expressed in state-space form. Therefore, as put forward, in the univariate SV model (see e.g. Kim and Stoffer, 2008 for the non periodic case and Boussaha and Hamdi, 2015 for the periodic case), alternative methods to the traditional Kalman filter, such as the sequential Monte Carlo ones, should be considered.

5.2. Maximum Likelihood based on the EM algorithm with particle filters and smoothers

We now describe another estimation approach which combines the particle filters and smoothers with expectation-maximization (EM) algorithm.

The particle filters are sequential Monte Carlo methods which can be applied to the general state-space models. These filters can be considered as a powerful alternative to the Kalman filter for the optimal estimation problems in a non-linear/non-Gaussian state-space framework (see e.g. Doucet and Johansen, 2011). Let $\underline{Y} = (X_0', X_1', ..., X_n', \varepsilon_1', \varepsilon_2', ..., \varepsilon_n')'$ and $\underline{X} = (X_0', X_1', ..., X_n')'$ denote the vector containing, respectively, the complete data and the log-volatilities data. For a given realization $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ of model (2), the complete log-likelihood function of the parameter vector θ can be expressed as follows:

$$
\begin{aligned}
-2\mathbf{L}(\theta; \underline{Y}) = C &+ \sum_{t=1}^{n} \log\left(\det\left(\Sigma_{\mathbf{e}}^{(t)}\right)\right) + \sum_{t=1}^{n} \log\left(\det\left(\Sigma_{\eta}^{(t)}\right)\right) + \sum_{t=1}^{n}\sum_{i=1}^{m} X_{t,i} \\
&+ \sum_{t=1}^{n} (X_t - \alpha_t - \Phi_t X_{t-1})' \left(\Sigma_{\mathbf{e}}^{(t)}\right)^{-1} (X_t - \alpha_t - \Phi_t X_{t-1}) \\
&+ \sum_{t=1}^{n} \varepsilon_t' diag\left(\exp\left\{-\frac{1}{2}X_t\right\}\right) \left(\Sigma_{\eta}^{(t)}\right)^{-1} diag\left(\exp\left\{-\frac{1}{2}X_t\right\}\right) \varepsilon_t,
\end{aligned}
$$

where C is a constant independent of θ. It is well known that, as in a variety of cases where the data is considered to be incomplete, the estimation problem can be easily done via the simple recursive EM algorithm introduced by Dempster et al. (1977). Recall that this algorithm works in an iterative way. In each iteration, there are two steps called E-step and M-step. To start the $(i+1)$-th iteration, we have the parameter estimation from the last iteration (or initial value) $\widehat{\theta}^{(i)}$, and we define the following Q-function in the E-step

$$\begin{aligned}
Q(\theta, \widehat{\theta}^{(i)}) &= \mathbb{E}\left[-2\mathbf{L}(\theta; \underline{Y})|Y, \widehat{\theta}^{(i)}\right] \\
&= C + \sum_{t=1}^{n} \log\left(\det\left(\Sigma_{\mathbf{e}}^{(t)}\right)\right) + \sum_{t=1}^{n} \log\left(\det\left(\Sigma_{\eta}^{(t)}\right)\right) + \sum_{t=1}^{n}\sum_{i=1}^{m} X_{t,i}^{(n)} \\
&+ \sum_{t=1}^{n} tr\left(\left(\Sigma_{\mathbf{e}}^{(t)}\right)^{-1} A_t\right) + \sum_{t=1}^{n} tr\left(\left(\Sigma_{\eta}^{(t)}\right)^{-1} B_t\right).
\end{aligned}$$

where

$$\begin{aligned}
A_t &= X_{t,t}^{(n)} - X_t^{(n)}\alpha_t' - X_{t,t-1}^{(n)}\Phi_t' - \alpha_t\left(X_t^{(n)}\right)' + \alpha_t\alpha_t' + \alpha_t\left(X_{t-1}^{(n)}\right)'\Phi_t' \\
&- \Phi_t X_{t-1,t}^{(n)} + \Phi_t X_{t-1}^{(n)}\alpha_t' + \Phi_t X_{t-1,t-1}^{(n)}\Phi_t', \\
B_t &= \mathbb{E}\left[diag\left(\exp\left\{-\frac{1}{2}X_t\right\}\right)\varepsilon_t\varepsilon_t' diag\left(\exp\left\{-\frac{1}{2}X_t\right\}\right)\middle| Y, \widehat{\theta}^{(i)}\right], \\
X_t^{(n)} &= \mathbb{E}\left(X_t|Y, \widehat{\theta}^{(i)}\right),
\end{aligned}$$

and

$$X_{t-h,t-k}^{(n)} = \mathbb{E}\left(X_{t-h}^{(n)}X_{t-k}'\middle| Y, \widehat{\theta}^{(i)}\right), \text{ for } h, k = 1, 0$$

Before going to the M-step, we need to evaluate the quantities $X_t^{(n)}$, $X_{t-h,t-k}^{(n)}$, A_t and B_t. As put forward, in the univariate AR-SV model (see Kim and Stoffer, 2008 and Boussaha and Hamdi, 2016), these quantities can be sequentially approximated in time by using the particle filtering and smoothing algorithms. The following is the algorithm for the filtering step, from which we will obtain M samples from the probability density function $p\left(X_t|\mathcal{F}_t\right)$, where $\mathcal{F}_t$ denotes the σ-algebra based on the information available up to time t.

Algorithm 1 Particle filter algorithm

1: Initialization: sample from $p_0(X_0)$ to obtain $f_0^{(j)}$ with initial weights $w_0^{(j)} = 1/M$, (for $j = 1, ..., M$), where M is the number of the particles.

2: Iterate, for $s = 1, ..., S$, and $\tau = 0, ..., \tau_1 - 1$.

a. For $j = 1, ..., M$

i. Simulate $\mathbf{e}_{s+\tau S}^{(j)} \sim \mathcal{N}(0, \Sigma_{\mathbf{e}}^{(s)})$.

ii. Compute $p_{s+\tau S}^{(j)} = \alpha_s + \Phi_s f_{s+\tau S-1}^{(j)} + \mathbf{e}_{s+\tau S}^{(j)}$.

iii. Evaluate importance weights: compute

$$\begin{aligned} w_{s+\tau S}^{(j)} &= w_{s+\tau S-1}^{(j)} p\left(\varepsilon_{s+\tau S} \left| p_{s+\tau S}^{(j)} \right.\right) \\ &\propto w_{s+\tau S-1}^{(j)} \\ &\times \exp\left\{-\frac{\left(diag\left\{\exp\left(-\frac{1}{2}p_{s+\tau S}^{(j)}\right)\right\}\varepsilon_{s+\tau S}\right)'\left(\Sigma_\eta^{(s)}\right)^{-1}\left(diag\left\{\exp\left(-\frac{1}{2}p_{s+\tau S}^{(j)}\right)\right\}\varepsilon_{s+\tau S}\right)}{2}\right\} \\ &\times \left(\det\left(\Sigma_\eta^{(s)}\right)\right)^{-1/2} \det\left(diag\left(\exp\left(-\frac{1}{2}p_{s+\tau S}^{(j)}\right)\right)\right). \end{aligned}$$

b. For $j = 1, ..., M$, normalize weights: compute

$$\widetilde{w}_{s+\tau S}^{(j)} = w_{s+\tau S}^{(j)} \Bigg/ \sum_{j=1}^{M} w_{s+\tau S}^{(j)} .$$

c. Compute the measure of degeneracy $n^{eff} = 1 \Big/ \sum_{j=1}^{M} \widetilde{w}_{s+\tau S}^{(j)2}$. If $n^{eff} \leq n^T$ (typically $n^T = M/2$) resample $\left\{\left(p_{s+\tau S}^{(j)}, \widetilde{w}_{s+\tau S}^{(j)}\right),\ j = 1, ..., M\right\}$ to obtain M equally-weighted particles $\left\{\left(f_{s+\tau S}^{(j)}, 1/M\right),\ j = 1, ..., M\right\}$.

3: Finally, the sequence of M particles $\left\{f_{s+\tau S}^{(j)}; j = 1, ..., M\right\}$ is a random sample from $p\left(X_{s+\tau S} | \mathcal{F}_{s+\tau S}\right)$, for $s = 1, ..., S$ and $\tau = 0, ..., \tau_1 - 1$.

Algorithm 2 Particle smoothing algorithm

1: For $j = 1, ..., M$, choose $s_n^{(j)} = f_n^{(i)}$, with probability $\widetilde{w}_n^{(i)}$. Let us fix $W_n^{(j)} = 1/M$.

2: For $j = 1, ..., M$.

a. For $s = S, ..., 1$ and $\tau = \tau_1 - 1, ..., 0$, calculate for $i = 1, ..., M$

$$\begin{aligned} W^{(i)}_{s+\tau S-1|\tau S+s} &= \widetilde{w}^{(i)}_{s+\tau S-1} \\ &\times \exp\left\{-\frac{\left(s^{(j)}_{s+\tau S}-\alpha_s-\Phi_s f^{(i)}_{s+\tau S-1}\right)'\left(\Sigma^{(s)}_{\mathbf{e}}\right)^{-1}\left(s^{(j)}_{s+\tau S}-\alpha_s-\Phi_s f^{(i)}_{s+\tau S-1}\right)}{2}\right\} \\ &\times \left(\det\left(\Sigma^{(s)}_{\mathbf{e}}\right)\right)^{-1/2}. \end{aligned}$$

b. For $i = 1, ..., M$, normalize the smoothed weights via

$$\widetilde{W}^{(i)}_{s+\tau S-1|s+\tau S} = W^{(i)}_{s+\tau S-1|s+\tau S} \bigg/ \sum_{j=1}^{M} W^{(i)}_{s+\tau S-1|s+\tau S}.$$

c. Draw $s^{(j)}_{s+\tau S-1}$from $f^{(i)}_{s+\tau S-1}$, with probability proportional to $\left\{\widetilde{W}^{(i)}_{s+\tau S-1|s+\tau S},\ i = 1, ..., M\right\}$.

3: Finally, compute $s = 1, ..., S$ and $\tau = 0, ..., \tau_1 - 1$,

$$\begin{aligned} \widehat{X}^{(n)}_{s+\tau S} &= \frac{\sum_{j=1}^{M} s^{(j)}_{s+\tau S}}{M}, \\ \widehat{P}^{(n)}_{s+\tau S} &= \frac{\sum_{j=1}^{M}\left(s^{(j)}_{s+\tau S}-\widehat{X}^{(n)}_{s+\tau S}\right)\left(s^{(j)}_{s+\tau S}-\widehat{X}^{(n)}_{s+\tau S}\right)'}{M-1}, \\ \widehat{P}^{(n)}_{s+\tau S,s+\tau S-1} &= \frac{\sum_{j=1}^{M}\left(s^{(j)}_{s+\tau S}-\widehat{X}^{(n)}_{s+\tau S}\right)\left(s^{(j)}_{s+\tau S-1}-\widehat{X}^{(n)}_{s+\tau S-1}\right)'}{M}, \end{aligned}$$

and

$$\widehat{B}_{s+\tau S} = \frac{\sum_{j=1}^{M} diag\left(\exp\left\{-\frac{s^{(j)}_{s+\tau S}}{2}\right\}\right)\varepsilon_{s+\tau S}\varepsilon'_{s+\tau S} diag\left(\exp\left\{-\frac{s^{(j)}_{s+\tau S}}{2}\right\}\right)}{M}.$$

After replacing $X_t^{(n)}$, $P_t^{(n)}$, $P_{t,t-1}^{(n)}$, A_t and B_t with their approximations, we thus go to the M-step, where the estimated parameter is obtained as follows

$$\widehat{\theta}^{(i+1)} = \arg\max_{\theta} Q\left(\theta, \widehat{\theta}^{(i)}\right).$$

The first derivatives of the Q-function with respect to θ are as follows

$$\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \alpha_s} = \left(\Sigma_{\mathbf{e}}^{(s)}\right)^{-1} \sum_{\tau=0}^{\tau_1 - 1} \left(2\alpha_s - 2X_{s+\tau S}^{(n)} + 2\Phi_s X_{s+\tau S-1}^{(n)}\right), \tag{8}$$

$$\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \left(vech\left(\Phi_s\right)\right)} = vech\left(\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \Phi_s}\right), \tag{9}$$

$$\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \left(vech\left(\Sigma_{\mathbf{e}}^{(s)}\right)\right)} = D_m' \times vec\left(\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \Sigma_{\mathbf{e}}^{(s)}}\right), \tag{10}$$

$$\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \left(vech\left(\Sigma_{\eta}^{(s)}\right)\right)} = D_m' \times vec\left(\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \Sigma_{\eta}^{(s)}}\right), \tag{11}$$

where

$$\begin{aligned}
\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \Phi_s} &= 2\left(\Sigma_{\mathbf{e}}^{(s)}\right)^{-1} \\
&\quad \times \sum_{\tau=0}^{\tau_1 - 1} \left[\alpha_s \left(X_{s+\tau S-1}^{(n)}\right)' - \left(X_{s+\tau S}^{(n)} \left(X_{s+\tau S-1}^{(n)}\right)' + P_{s+\tau S, s+\tau S-1}^{(n)}\right)\right. \\
&\quad \left. + \Phi_s \left(X_{s+\tau S-1}^{(n)} \left(X_{s+\tau S-1}^{(n)}\right)' + P_{s+\tau S-1}^{(n)}\right)\right], \\
\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \Sigma_{\mathbf{e}}^{(s)}} &= \left(\Sigma_{\mathbf{e}}^{(s)}\right)^{-1} \sum_{\tau=0}^{\tau_1 - 1} \left[\mathbf{I}_m - A_{s+\tau S}\left(\Sigma_{\mathbf{e}}^{(s)}\right)^{-1}\right], \\
\frac{\partial Q(\theta, \widehat{\theta}^{(i)})}{\partial \Sigma_{\eta}^{(s)}} &= \left(\Sigma_{\eta}^{(s)}\right)^{-1} \sum_{\tau=0}^{\tau_1 - 1} \left[\mathbf{I}_m - B_{s+\tau S}\left(\Sigma_{\eta}^{(s)}\right)^{-1}\right],
\end{aligned}$$

and where D_m denotes the $\left(m^2 \times \frac{1}{2}m\left(m+1\right)\right)$ duplication matrix such that $vec\left(A\right) = D_m \times vech\left(A\right)$, for any $\left(m \times m\right)$-matrix A (see e.g. Lütkepohl, 1996, Section 9.5). Here $vech$ is the half column staking operator.

The estimates of the parameters $\theta^{(i+1)}$ can then be obtained by equating (8), (9), (10) and (11) to zero. Hence, at the $(i+1)$-th iteration, the parameter estimates of α_s, $vech\left(\Phi_s\right)$, $vech\left(\Sigma_{\mathbf{e}}^{(s)}\right)$ and $vech\left(\Sigma_{\eta}^{(s)}\right)$, for $s=1,...,S$, are summarized below

$$\begin{cases} vech\left(\widehat{\Sigma}_{\eta}^{(s)}\right)=\frac{1}{\tau_1}\sum_{\tau=0}^{\tau_1-1} vech\left(\widehat{B}_{s+\tau S}\right), \\ vech\left(\widehat{\Phi}_s\right)=\left[L_m\left\{\left(\frac{1}{\tau_1}\left(\sum_{\tau=0}^{\tau_1-1}X_{s+\tau S-1}^{(n)}\right)\left(\sum_{\tau=0}^{\tau_1-1}X_{s+\tau S-1}^{(n)}\right)'\right.\right.\right. \\ \qquad\left.\left.\left.-\sum_{\tau=0}^{\tau_1-1}\left(X_{s+\tau S-1}^{(n)}\left(X_{s+\tau S-1}^{(n)}\right)'+P_{s+\tau S-1}^{(n)}\right)\right)\otimes\mathbf{I}_m\right\}L_m'\right]^{-1} \\ \qquad\times vech\left(\frac{1}{\tau_1}\left(\sum_{\tau=0}^{\tau_1-1}X_{s+\tau S}^{(n)}\right)\left(\sum_{\tau=0}^{\tau_1-1}X_{s+\tau S-1}^{(n)}\right)'\right. \\ \qquad\left.-\sum_{\tau=0}^{\tau_1-1}\left(X_{s+\tau S}^{(n)}\left(X_{s+\tau S-1}^{(n)}\right)'+P_{s+\tau S,s+\tau S-1}^{(n)}\right)\right), \\ \widehat{\alpha}_s=\frac{1}{\tau_1}\sum_{\tau=0}^{\tau_1-1}X_{s+\tau S}^{(n)}-\frac{1}{\tau_1}\widehat{\Phi}_s\sum_{\tau=0}^{\tau_1-1}X_{s+\tau S-1}^{(n)}, \\ vech\left(\widehat{\Sigma}_{\mathbf{e}}^{(s)}\right)=\frac{1}{\tau_1}\sum_{\tau=0}^{\tau_1-1}vech\left(\widehat{A}_{s+\tau S}\right), \end{cases}$$

where L_m denotes the $\left(\frac{1}{2}m\left(m+1\right)\times m^2\right)$ elimination matrix such that

$$vech\left(A\right)=L_m\times vec\left(A\right),$$

for any $(m\times m)$-matrix A (see e.g. Lütkepohl, 1996, Section 9.6).

6. Application

This final section provides an empirical application. More precisely, we first describe our data and provide some preliminary analysis in order to justify the use of periodic and multivariate models. We thus apply the theoretical analysis developed in Section 2 to provide a multivariate modeling of oil spot price and three exchange rates time series.

6.1. Data and preliminary analysis

We model four time series: the daily Saharan Blend oil (SB) spot price time series (provided by the Algerian Ministry of Energy) and three daily time series of exchange rates: Euro/U.S. dollar (EUR/USD) provided by Thomson Reuters, Euro/Algerian dinar (EUR/DZD) and U.S. dollar/ Algerian dinar (USD/DZD) provided by the Bank of Algeria. We first removed all the days in which the SB market was closed. The period of analysis is 3 January 2005 to 31 December 2015. The final sample consists of 2836 observations for each series.

In this empirical study, the model is structured as follows. For notational purposes, we collect the original Saharan Blend oil price and three exchange rates at time t in the vector

$$Y_t = \begin{pmatrix} Y_{t,1} \\ Y_{t,2} \\ Y_{t,3} \\ Y_{t,4} \end{pmatrix} = \begin{pmatrix} SB_t \\ EUR/USD_t \\ EUR/DZD_t \\ USD/DZD_t \end{pmatrix},$$

and their corresponding log-return series in the vector $Z_t = (Z_{t,1}, Z_{t,2}, Z_{t,3}, Z_{t,4})'$, where $Z_{t,i} = \log(Y_{t,i}) - \log(Y_{t-1,i})$, for $i = 1, 2, ..., 4$.

The original series $Y_{t,i}$ and the log-return series $Z_{t,i}$ are respectively shown in Figures 1 and 2. It turns out that all the original series $Y_{t,i}$ are nonstationary while the log-return series appear to be stationary. Moreover, the volatility clustering phenomenon is clearly evident in each of them. Not surprisingly, the volatility was higher during the crisis in 2008 and after 2015. The volatilities in the log-returns series are subject to some breaks: high volatility between April 2008 and July 2009 and after 2015 and low volatility between December 2006 and March 2008 and between August 2012 and December 2014. Additionally, in the whole study period the SB log-return is the most volatile, followed by the EUR/USD and EUR/DZD log-returns. The USD/DZD log-return is the most stable one.

Some descriptive statistics are provided for the four log-return series in Tables 1-4. These statistics are very similar across these series. Their standard deviations are much greater than the means in absolute value, indicating that the means are not significantly different from zero. Moreover, their excess kurtosis is significantly positive. Therefore, these series have heavy tails relative to the normal distribution, which is also typical in these financial data. The Jarque-Bera test rejects the normality of the four log-return distributions.

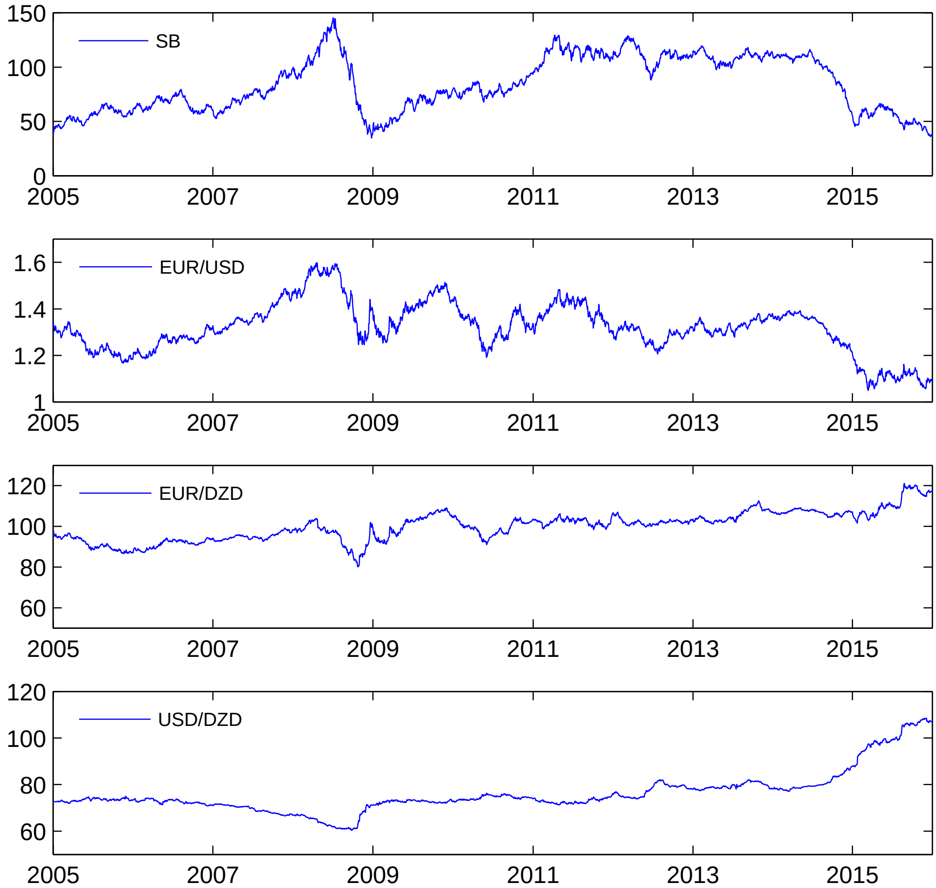

Figure 1: The original series.

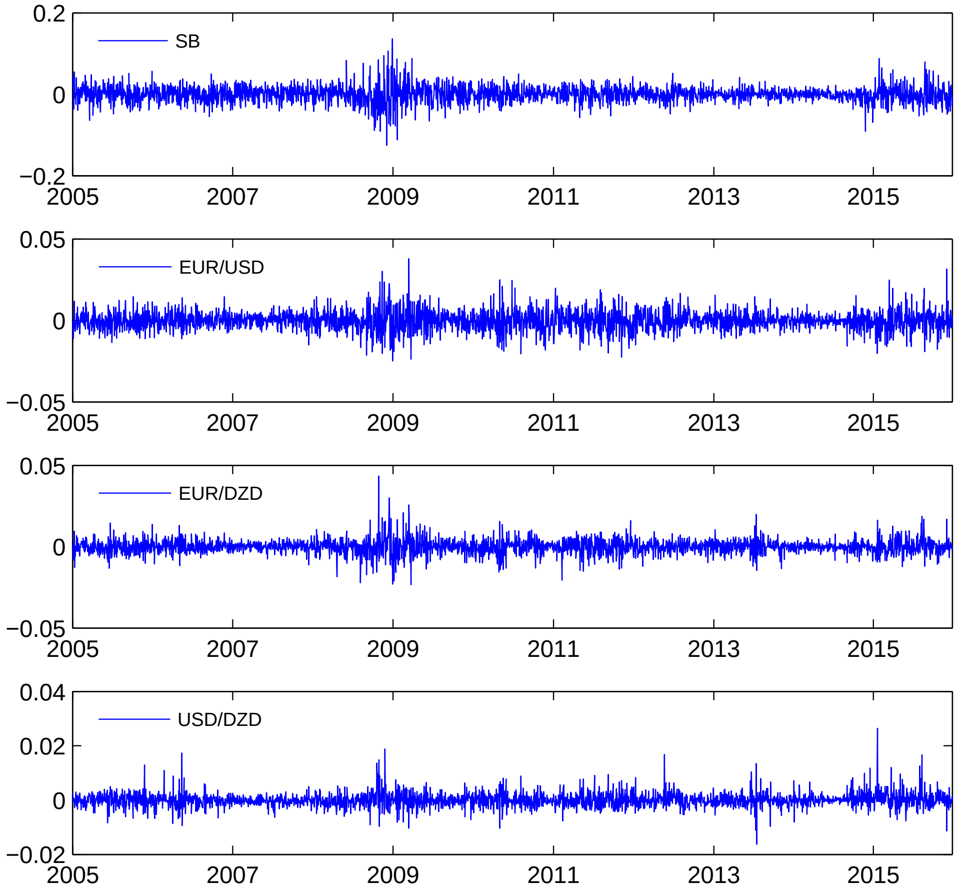

Figure 2: The log-return series.

Table 1. Descriptive statistics of SB log-returns for each day of the week as well as the entire study period

	Monday	Tuesday	Wednesday	Thursday	Friday	All days
# of obs.	567	570	572	569	557	2835
Mean	−0.0014	0.0000	0.0005	0.0011	−0.0004	-2.29×10^{-5}
Median	0.0000	0.0000	0.0011	0.0020	−0.0003	9.32×10^{-5}
Maximum	0.1367	0.0633	0.0850	0.1067	0.0825	0.1367
Minimum	−0.0777	−0.1119	−0.0812	−0.0909	−0.1244	−0.1244
Std. Dev.	0.0212	0.0199	0.0182	0.0211	0.0193	0.0110
Skewness	0.7466	−0.4172	−0.0594	0.0078	−0.3224	0.0276
Kurtosis	8.5317	5.4482	5.0156	5.9985	7.9584	6.7408
Jarque-Bera	775.5827	158.8808	97.1668	213.1664	580.2407	1653.3850

Table 2. Descriptive statistics of EUR/USD log-returns for each day of the week as well as the entire study period

	Monday	Tuesday	Wednesday	Thursday	Friday	All days
# of obs.	567	570	572	569	557	2835
Mean	−0.0005	−0.0003	0.0000	0.0002	0.0001	-7.56×10^{-5}
Median	−0.0001	−0.0001	0.0000	0.0001	0.0002	0.0000
Maximum	0.0236	0.0264	0.0272	0.0381	0.0301	0.0381
Minimum	−0.0332	−0.0213	−0.0206	−0.0254	−0.0212	−0.0332
Std. Dev.	0.0072	0.0058	0.0063	0.0065	0.0067	0.0065
Skewness	−0.2244	0.2663	0.1105	0.2828	0.1631	0.0833
Kurtosis	4.3099	5.6897	4.5673	5.9422	4.5940	5.0206
Jarque-Bera	45.2972	178.5543	59.7092	212.8093	61.4376	485.5663

Table 3. Descriptive statistics of EUR/DZD log-returns for each day of the week as well as the entire study period

	Monday	Tuesday	Wednesday	Thursday	Friday	All days
# of obs.	567	570	572	569	557	2835
Mean	−0.0002	−0.0002	0.0003	0.0003	0.0001	6.32×10^{-5}
Median	−0.0001	0.0001	0.0002	0.0004	0.0000	0.0001
Maximum	0.0209	0.0170	0.0313	0.0497	0.0226	0.0497
Minimum	−0.0233	−0.0230	−0.0158	−0.0195	−0.0220	−0.0233
Std. Dev.	0.0049	0.0041	0.0044	0.0049	0.0048	0.0046
Skewness	−0.0819	−0.4649	0.8022	2.2173	0.1346	0.6007
Kurtosis	6.2042	6.6266	8.9065	25.1483	6.8762	11.6985
Jarque-Bera	243.1895	332.8976	892.8278	12096.3000	350.3907	9108.4040

Table 4. Descriptive statistics of USD/DZD log-returns for each day of the week as well as the entire study period

	Monday	Tuesday	Wednesday	Thursday	Friday	All days
# of obs.	567	570	572	569	557	2835
Mean	0.0003	0.0000	0.0000	0.0001	0.0003	0.0001
Median	0.0001	0.0000	−0.0001	−0.0001	0.0001	-1.89×10^{-5}
Maximum	0.0282	0.0094	0.0138	0.0181	0.0335	0.0335
Minimum	−0.0092	−0.0113	−0.0070	−0.0100	−0.0132	−0.0132
Std. Dev.	0.0030	0.0023	0.0024	0.0028	0.0033	0.0028
Skewness	1.8854	0.0974	1.0514	1.0981	2.8623	1.8005
Kurtosis	17.9821	6.3905	8.5105	9.5237	27.1108	18.9315
Jarque-Bera	5638.8570	273.9248	829.1143	1123.3530	14252.2300	31513.4400

Table 5 provides the sample correlation matrix for the entire study period. Correlation coefficients vary from −0.5591 to 0.6720. The strongest negative correlation is for the pair (EUR/USD, USD/DZD) and the strongest positive correlation is for the pair (EUR/USD, EUR/DZD). The SB is the least correlated with all the exchange rates. This suggests that estimating a joint model may yield interesting information on the relationships between our data series. This is why we do use, in the sequel, a multivariate modelling of the collected data.

Table 5. Correlation matrix of log returns

log-returns	SB	EUR/USD	EUR/DZD	USD/DZD
SB	1	0.0491	0.0926	−0.1260
EUR/USD		1	0.6720	−0.5591
EUR/DZD			1	−0.4003
USD/DZD				1

Tables 1-4 also report the descriptive statistics of the four log-returns for each day of the week. The findings indicate that for the USD/DZD (resp. EUR/DZD, EUR/USD, SB), the lowest (−0.0132) (resp. −0.0233, −0.0332, −0.1244) and the highest (0.0335) (resp. 0.0497, 0.0381, 0.1367) returns are observed in Friday (resp. Friday and Thursday, Monday and Thursday, Friday and Monday). For the analyzed period, descriptive statistics per day register the lowest standard deviation of 0.0023 (resp. 0.0041, 0.0058, 0.0182) in Tuesday (resp. Tuesday, Tuesday, Wednesday) for the USD/DZD log-returns series (resp. EUR/DZD, EUR/USD, SB), while the highest standard deviation of 0.0033 (resp. 0.0049, 0.0072, 0.0212) is observed for Friday (resp. Monday and Thursday, Monday, Monday). The skewness for all the days of the week and for all log-return series is positive, while it is negative for Monday and Tuesday in EUR/DZD, for Monday in EUR/USD and for Tuesday, Wednesday and Friday in SB log-returns. Furthermore, one of the features which prominently stands out most from Tables 1-4 is that the kurtosis for each day-of-the-week is much larger than 3. This reflects the fact that the tails of the distributions of all the analyzed log-returns per day are fatter than the tails of the normal distribution.

Table 6. Periodic autocorrelations of log-return series

log-returns	Monday	Tuesday	Wednesday	Thursday	Friday
SB	0.0565	-0.0383	0.0252	0.0596	0.0241
EUR/USD	0.0272	-0.1629	0.0182	-0.0006	-0.0339
EUR/DZD	0.0424	-0.0158	0.0231	0.2094	0.0877
USD/DZD	0.1617	-0.0617	-0.0685	0.1639	0.1728

These descriptive statistics of the four log-returns for each day of the week are not surprising since it is usually observed in financial time series where correlations between daily returns and volatility display some day-of-the-week effects (see e.g. Franses and Paap, 2000; Hamdi and Souam, 2017). Such a property may be useful as sometimes one may expect economic agents to have different behavior in different days of the week. Indeed, from daily first-order autocorrelations for our return series (see Table 6), it is evident that returns on Mondays are positively correlated with those on the preceding Fridays, while returns on Tuesdays are negatively correlated with those on Mondays. This is why we do consider the periodicity as an important issue in our modelling of these time series.

6.2. Empirical results

In the following, we propose a periodic model that allows for the description of the day-of-the-week or intraweek effect in the daily exchange rate and oil price series. Such a formulation with period $S = 5$ seems to be well adapted to explain the presence (or not) of intraweek seasonality as have done Franses and Paap (2000) for modelling day-of-the-week seasonality in the $S\&P$ 500 index through a univariate periodic $AR\text{-}GARCH$ process. We will consider the 5-periodic $PVAR\text{-}SV$ model, where the parameters are allowed to vary with the day of the week ($s = 1, 2, 3, 4$ and 5)

$$s = \begin{cases} 1 & \text{if the day corresponding is a Monday,} \\ 2 & \text{if the day corresponding is a Tuesday,} \\ \vdots & \\ 5 & \text{if the day corresponding is a Friday.} \end{cases}$$

From Table 6, it is clear that the series Z_t, $t = 1, ..., 2835$, exhibits a periodicity in the conditional mean. In order to capture such a periodicity, which is not possible by the use of a pure *PVAR-SV* model, we have applied a multivariate periodic autoregressive (*PVAR*) filter to Z_t and we have considered the residuals as the underlying series for our *PVAR-SV* modeling. Table 7 shows the results of the application of the filter.

The plot of the real series $Z_{t,i}$, $i = 1, ..., 4$, the adjusted and the residuals ones computed from the $PVAR(2)$ fitted model are given by Figure 3. As a second step, we use these residuals to estimate the volatility.

The estimated parameters of the 5-periodic *PVAR-SV* model are presented in Table 8. We observe that all the log-return series examined exhibit highly persistent volatilities. The persistent coefficient ranges from $\prod_{s=1}^{S} \phi_{i,i}^{(s)} = 0.7960$ for USD/DZD to 0.7543 for SB. This allows us to conclude that the *PVAR-SV* estimated model is periodically stationary as the quantity characterizing the estimated processes, namely $\rho\left(\prod_{s=1}^{S} \Phi_s\right)$ defined in (5), is smaller than unity ($\rho\left(\prod_{s=1}^{S} \Phi_s\right) = 0.7960$). On the other hand, when we analyze the estimated model period by period, we notice that only the models corresponding to Wednesday and Thursday are nonstationary. Indeed, we have $\rho(\Phi_s)$ equal, respectively, to 0.9971, 0.9571, 1.5110, 1.1020 and 0.9340 for $s = 1, 2, 3, 4$ and 5.

These results are consistent with the fact that exchange rates and oil price volatilities are often clustered. In all cases, the transmission coefficients ($\phi_{i,j}^{(s)}$) are quite weak and therefore there is a volatility interaction in daily data. For instance, from the estimated model, information in oil market is quickly transmitted and incorporated into the EUR/USD, EUR/DZD and USD/DZD exchange rates. This could suggest that volatility in the oil market have significant impacts on the FX market, especially on Algerian dinar.

Another interesting phenomenon appears in our empirical analysis. It deals with the fact that the seasonal return shock correlation $\frac{\Sigma_\eta^{(s)}(i,j)}{\sqrt{\Sigma_\eta^{(s)}(i,i)}\sqrt{\Sigma_\eta^{(s)}(j,j)}}$ ($s = 1, ..., 5$ and $i, j = 1, ..., 4$) is significantly negative between USD/DZD and the other series, while it is significantly positive in all the other cases (between EUR/USD and SB and between EUR/DZD and SB and EUR/USD).

Figure 5 shows the estimated volatilities computed from 5-periodic *PVAR-SV* model of SB, EUR/USD, EUR/DZD and USD/DZD. We can observe that the estimated volatilities reflect quite well the variation of the residuals. Furthermore, from Table 9 it turns out that the empirical coverages of the *PVAR-SV*-based prediction intervals are closer to the nominal coverages. These results show that the *PVAR-SV* model, fitted to the residuals computed from *PVAR* modeling of our daily log-return time series, seems well accurate and has a well volatility forecasting performance.

Table 7. $PVAR$ filter to $Z_t : Z_{s+5\tau} = \mu_s + \Psi_{s,1} Z_{s+5\tau-1} + \Psi_{s,2} Z_{s+5\tau-2} + \varepsilon_{s+5\tau}$.

	μ_s	$\Psi_{s,1}$				$\Psi_{s,2}$			
$s=1$	0	0	0	0	0	0	0	0	0
	0	0.0504	−0.3832	0.5118	−0.5567	0	0	0	−0.4004
	0	0	0	0	0	0	0	0	0
	0.0003	0	0	0	0.1306	0	0	0	0.1020
$s=2$	0	0	0	0	0	0	0	0	0
	0	0.0495	−0.6245	0.5911	−0.5205	0	−0.2530	0.2387	−0.4112
	0	0	0	0	0	0	0	0	0
	0	−0.0162	0.0437	0	0	0	0	0	0
$s=3$	0	0	0	0	0	0	0	0	0
	0	0.0495	−0.5059	0.6890	−0.3989	0.0309	−0.2503	0.3707	0
	0	0.0217	0	0	0	0.0219	0	0.1018	0.3009
	0	−0.0118	0	0	0	0	0	0	0
$s=4$	0	0	0	0	0	0	0	0	0
	0	0.0496	−0.4237	0.5522	−0.4355	0	−0.1685	0.3661	0
	0	0.0489	0	0.2837	0.4225	0	0	0.1323	0.2303
	0	−0.0155	0	0.1460	0.2786	0	0	0	0
$s=5$	0	0	0.5124	−0.4541	0	0	0	0	0
	0	0.0566	−0.4402	0.3986	−0.6652	0	−0.1508	0.3265	0
	0	0.0236	0.1061	0	0.2977	0	0	0	0
	0	0	0	0.3098	0.4023	0	0	−0.0876	−0.1330

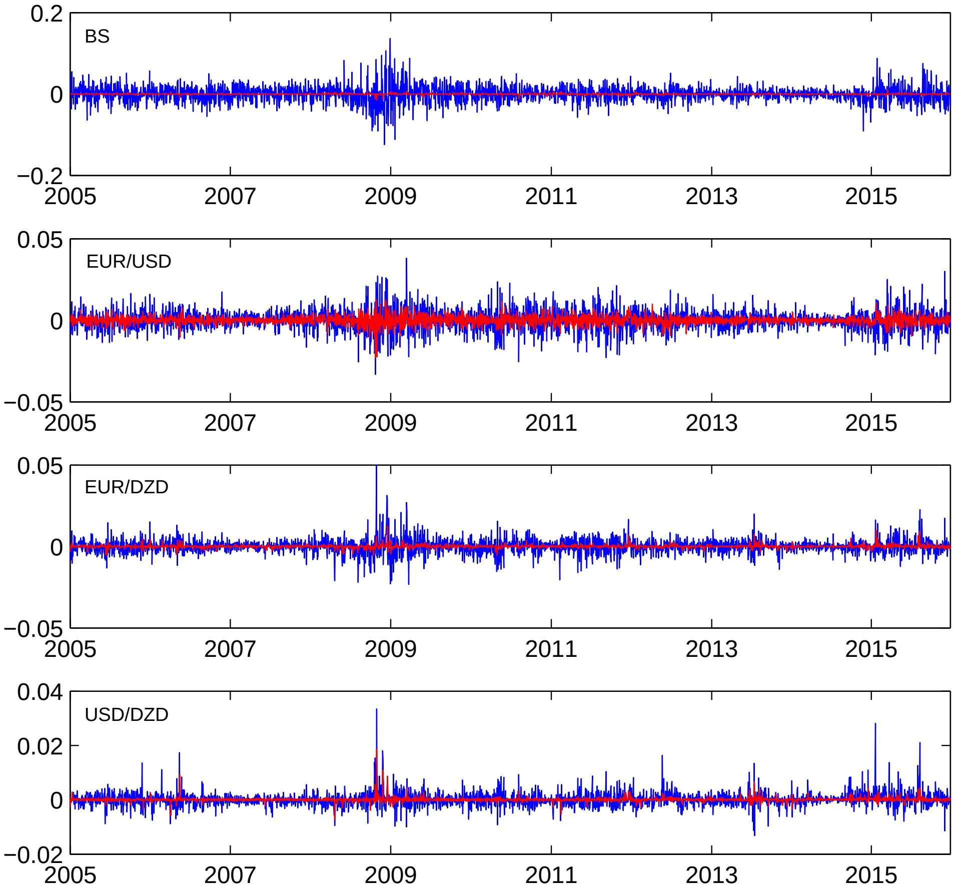

Figure 3: Real and estimated series from $PVAR_5(2)$ model.

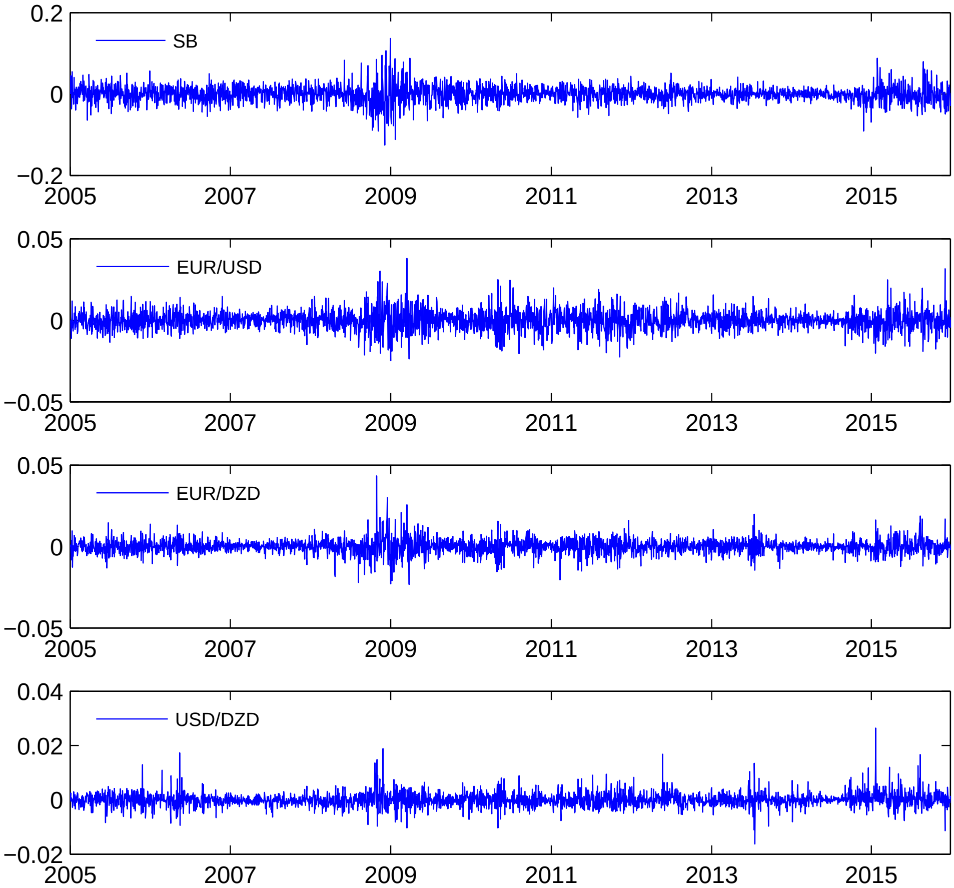

Figure 4: The residual series computed from $PVAR_5(2)$ model.

Table 8. Parameter estimates of 5-periodic *PVAR-SV* model for $\varepsilon_{s+5\tau} : \varepsilon_{s+5\tau} = V_{s+5\tau}^{1/2}\eta_{s+5\tau}$, $X_{s+5\tau} = \alpha_s + \Phi_s X_{s+5\tau-1} + \mathbf{e}_{s+5\tau}$, $V_t = diag\left(\exp X_t\right)$, $\eta_{s+5\tau} \sim \mathcal{N}\left(0_{4\times1}, \Sigma_\eta^{(s)}\right)$ and $\mathbf{e}_{s+5\tau} \sim \mathcal{N}\left(0_{4\times1}, \Sigma_{\mathbf{e}}^{(s)}\right)$

	α_s	Φ_s			
$s=1$	−0.20262	0.98040			
	0.01157	0.00043	0.99685		
	0.00686	0.00024	−0.00058	0.99705	
	0.00704	0.00038	0.00067	−0.00015	0.99605
$s=2$	−1.45498	0.83309			
	−1.20156	0.00423	0.95710		
	−0.86349	0.03949	0.02311	0.94923	
	−0.61738	0.00272	0.05241	0.04000	0.94269
$s=3$	3.91287	1.51103			
	−0.00441	0.00338	0.91994		
	−0.05308	0.00181	−0.00097	0.91795	
	−0.01491	0.00031	0.00238	0.00040	0.91900
$s=4$	0.69615	1.10199			
	−0.84999	−0.01508	0.97555		
	−0.55723	0.03054	0.00381	0.96981	
	−0.19088	0.00130	0.03733	0.01311	0.98764
$s=5$	−3.17158	0.55463			
	−0.50141	−0.00491	0.91828		
	−0.28571	0.00438	0.01226	0.92180	
	−0.07079	0.00514	0.01030	0.01192	0.93396

Table 8. (Continued.)

	$\Sigma_{\eta}^{(s)}$	$\Sigma_{\mathbf{e}}^{(s)} \times 10^{-3}$
$s=1$	$\begin{pmatrix} 0.74094 & & & \\ 0.01565 & 0.93385 & & \\ 0.08498 & 0.62127 & 0.90496 & \\ -0.08795 & -0.63367 & -0.51178 & 0.91656 \end{pmatrix}$	$\begin{pmatrix} 0.00001 & & & \\ 0.00001 & 0.09361 & & \\ 0.00001 & -0.00100 & 0.09663 & \\ 0.00001 & -0.00029 & 0.00194 & 0.09346 \end{pmatrix}$
$s=2$	$\begin{pmatrix} 0.89185 & & & \\ 0.05403 & 1.09090 & & \\ 0.09130 & 0.77576 & 1.29935 & \\ -0.13203 & -0.69250 & -0.80344 & 1.30805 \end{pmatrix}$	$\begin{pmatrix} 0.00001 & & & \\ 0.00229 & 131.77359 & & \\ -0.00209 & 12.19494 & 124.73988 & \\ 0.00239 & 10.40001 & 10.84510 & 127.80300 \end{pmatrix}$
$s=3$	$\begin{pmatrix} 0.89839 & & & \\ 0.07722 & 0.68868 & & \\ 0.10155 & 0.49622 & 0.69023 & \\ -0.10294 & -0.39278 & -0.34332 & 0.66463 \end{pmatrix}$	$\begin{pmatrix} 0.54644 & & & \\ 0.02714 & 2.54788 & & \\ 0.02448 & -0.06642 & 2.54067 & \\ 0.02688 & -0.06561 & -0.02913 & 2.49501 \end{pmatrix}$
$s=4$	$\begin{pmatrix} 1.15961 & & & \\ 0.02547 & 1.17425 & & \\ 0.03613 & 0.78995 & 1.10038 & \\ -0.10227 & -0.82562 & -0.71740 & 1.26023 \end{pmatrix}$	$\begin{pmatrix} 224.15549 & & & \\ 7.38600 & 73.34013 & & \\ -0.13731 & 7.59376 & 66.03193 & \\ 6.12611 & 4.87846 & 2.96769 & 68.00796 \end{pmatrix}$
$s=5$	$\begin{pmatrix} 0.64827 & & & \\ 0.06520 & 0.94032 & & \\ 0.07977 & 0.70570 & 0.97110 & \\ -0.10650 & -0.64738 & -0.57424 & 1.06039 \end{pmatrix}$	$\begin{pmatrix} 57.78523 & & & \\ 1.85298 & 38.40728 & & \\ 2.47936 & 2.24313 & 37.65754 & \\ 0.28442 & -0.23460 & 1.51726 & 35.59162 \end{pmatrix}$

Table 9. Empirical coverages of the $(1-\alpha)100\%$ one-step-ahead prediction intervals

	$(1-\alpha)100\%$						
	50%	60%	70%	80%	90%	95%	99%
Saharan Blend oil prices	53.81	62.50	71.61	79.80	87.96	92.69	96.96
Euro/U.S. dollar	53.25	62.96	72.60	81.04	89.65	93.33	97.78
Euro/Algerian dinar	52.37	61.62	70.37	79.41	87.71	92.02	96.36
U.S. dollar/Algerian dinar	53.50	62.61	71.68	79.31	87.25	91.56	95.97

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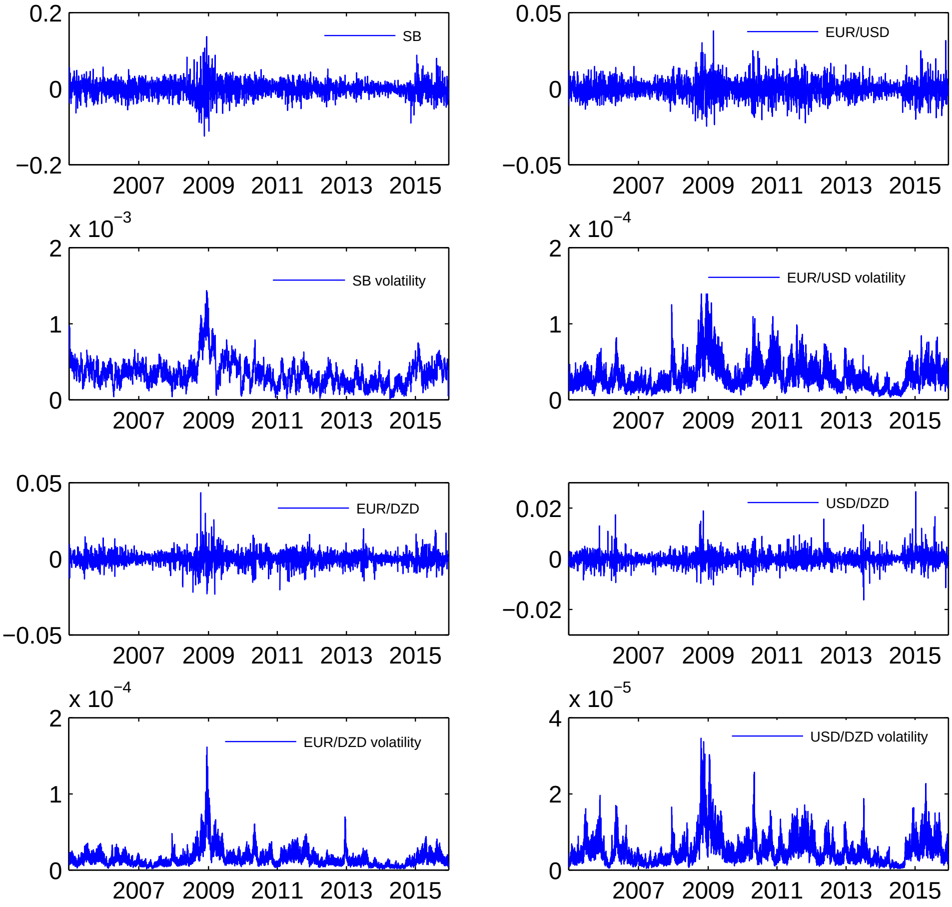

Figure 5: Estimated volatility from $PVAR\text{-}SV_5$ model.

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