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The Ricardian Dynamics Revisited

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Abstract. The Ricardian dynamics describe the substitution of a new marginal method for an outgoing marginal method when demand increases. The process of extension or intensification of cultivation allows for spasmodic changes in prices and rents but is smooth on the physical side. We criticize the notion of extension of cultivation and show the existence of non-Ricardian equilibria, when some good is not produced by a marginal method. The working of the dynamics requires that the incoming method satisfies a productivity condition which is expressed in algebraic terms. A parallel is drawn between Ricardo's views on dynamics and the working of a Lemke algorithm for linear complementarity problems.

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1 Introduction

The[1] Ricardian dynamics are among the most classical pieces of economic analysis: Ricardo (1817) combined parsimonious assumptions with a deep analysis of the extension of cultivation, from which he drew dramatic conclusions on the evolution of distribution between rents and profits and a plea for free trade. It is often admitted that, under the retained hypotheses, which ignore technical progress, the description of the extension or intensification of cultivation is correct and general (the argument is used today to explain the rise of the prices of raw materials). We intend to show that the dynamics are less obvious than it may appear and that their working requires a productivity condition which does not hold automatically.

Let us start from a long-term equilibrium for a given demand. Somewhat paradoxically, Ricardo's strategy consists in eliminating rents in the first step of his analysis. This is done by considering the operated industrial methods and the marginal agricultural methods, which pay a zero rent. For a given distribution variable (which is the real wage in Ricardo's analysis), these methods determine the set of prices, as if we were contemplating an economic system with no scarce resource. Ricardo eventually reintroduces rents as a reflection of differential costs between lands or agricultural methods. The increase in the demand for agricultural products leads to a change of equilibrium and triggers the dynamics. As long as lands of the same grade are available, the extension of cultivation requires an adjustment of activity levels but no changes in prices, wages and rents. When some land becomes fully cultivated, a physical limit is reached and the price of the scarce agricultural good rises suddenly up to the level where the cultivation of a more costly type of land becomes profitable. Then the previous marginal land yields a positive rent and all prices are modified. A new equilibrium is found and the transformation opens another quiet period with adaptations of activity levels, followed again by a new rupture. We consider as an essential feature of Ricardo's views that the spasmodic upheavals in value and distribution do not prevent a smooth physical transition from the previous to the new long-term equilibrium: in normal times, some methods of cultivation are progressively extended; from time to time, the introduction of cultivation on a new land, that the economists consider as a break, is hardly noticeable for a geographer.

The paper studies the Ricardian dynamics in a general framework with several agricultural goods. Curiously enough, the topic has rarely been examined in the analytical literature. It will turn out that the traditional de-

[1] With acknowledgements to Guido Erreygers for many discussions on the topic.

scription, which argues that the demand for corn triggers the introduction of a new corn method on another field (extension of cultivation) or that of more productive corn method on the same field (intensification of cultivation), is a too much simplified story. Section 2 shows that the equilibria may be of an unexpected (non-Ricardian) type. Section 3 analyzes the transition between consecutive equilibria and establishes a criterion for the working of the dynamics. Section 4 draws a parallel between the Ricardian dynamics and the Lemke (1965) algorithm, initially conceived to find a solution to bimatrix games. Some references are given in Section 5.

The Ricardian dynamics result from a comparison of long-term equilibria: each equilibrium is associated with a given demand for industrial and agricultural goods and the rates of profits are uniform. We shall not consider time explicitly and, when demand changes, disequilibria will only be referred to as the reason for a move from an equilibrium to another. Following Sraffa's (1960) proposal, we assume that the independent distribution variable is the given rate of profits rather than the real wage: this inessential change with regard to Ricardo's own hypothesis allows for analytical simplifications. Finally, since our basic purpose is to clarify the Ricardian dynamics, we discard any type of degeneracy and reduce calculations to minimum.

2 Non-Ricardian equilibria

Let there be a multisector reproduction model with n industrial or agricultural commodities. All methods are of the single-product type with constant returns (land is not treated here as a joint product of agriculture), but each agricultural method admits an upper activity level due to the scarcity of lands. In a steady state, the industrial methods yield the ruling rate of profits r (r is nonnegative and economically feasible) if they are operated, less if they are non operated. As for the agricultural methods, the rent is zero on partially cultivated lands (marginal land) and nonnegative (in fact positive, flukes apart) on fully cultivated lands: the positivity of rent stems from the scarcity of a given grade of land. Its level reflects the differences in the costs of production for two methods producing the same good and compensates for it: this differential cost may concern two methods of cultivation on different lands (extensive rent) or two methods of cultivation on the same land, one of them being more productive per acre but more expensive per quarter of product than the other (intensive rent). Once rent is taken into account, each good has a unique price and the rates of profit are uniform among all industries. Some lands are not cultivated at all, because this would be too costly.

Let us start from an initial long term-equilibrium I sustained by the price vector p_I (labor is chosen as numéraire) and assume that the demand for corn increases. An adaptation of activity levels, with no change in prices and rents, is a sufficient response of the productive system until some marginal land becomes fully cultivated. When a limit is reached, the price of corn rises, and Ricardo assumes that it increases up to the level where a new corn method becomes profitable, be it on a new type of land (extension of cultivation) or an already cultivated land (intensification of cultivation): a new equilibrium II with higher prices p_{II} and higher rents succeeds equilibrium I and provides a solution to the scarcity problem.

The point missed by Ricardo is that the rise in the price of corn triggers a rise in the prices of all commodities, be they industrial (iron) or agricultural (rice), in the production of which corn enters as a direct or indirect input. It may well be the case that a new iron method or a new rice method becomes profitable before any new corn method. Imagine for simplicity that, when the price of corn changes, the adjustment of the other prices to the changing costs is immediate, so that the market has only to find the new price of corn by means of a sequence of small successive rises. In that process, if an iron method which was not profitable at the initial price vector becomes profitable before any new corn method, that iron method would pay extra-profits if the rise continued, and this would be incompatible for the notion of long-equilibrium for an industrial method. If it is a rice method on a new land which becomes first profitable, the extra-profits might be absorbed and transformed into a rent paid to the landowner, but a comparison of the physical sides of both equilibria shows that a land which was not exploited at all at the first equilibrium would be fully cultivated at the second equilibrium, this being a condition for reaping a positive rent. The notion of Ricardian dynamics excludes such a discontinuity in activity levels. To sum up, the answer of the market to a scarcity of corn may consist in introducing a new iron method or a new rice method instead of a new corn method, as this may be the only way to ensure a smooth physical transition from the old to the new equilibrium.

A numerical example illustrates the point. For simplicity the rate of profits is zero (the phenomenon we point at occurs even under the golden rule hypothesis), but little would be changed for a small positive rate of profits.

Example 1. The rate of profits is zero. Let there be $n = 2$ agricultural goods, corn and rice, and $l = 4$ types of lands. Lands 1 and 2 are specialized in the production of corn, lands 3 and 4 in that of rice, with a unique method on each type of land (extensive cultivation). The available area of land i is denoted by $\overline{\Lambda_i}$. Per unit of land, the constant returns methods are described as follows:

Corn methods:

(method 1:) 1 qr. corn + 2 qr. rice + 4 labour + 1 $\text{land}_1 \rightarrow$ 7 qr. corn ($\overline{\Lambda_1} = 1$)

(method 2:) 1 qr. corn + 2 qr. rice + 4 labour + 1 $\text{land}_2 \rightarrow$ 2 qr. corn

Rice methods:

(method 3:) 2 qr. corn + 1 qr. rice + 4 labour + 1 $\text{land}_3 \rightarrow$ 7 qr. rice ($\overline{\Lambda_3} = 2$)

(method 4:) 1 qr.corn + 3 qr. rice + 4 labour + 1 $\text{land}_4 \rightarrow$ 7 qr. rice ($\overline{\Lambda_4} > 1.5$)

Let the final demand basket be $d(t) = (t, 4)$. When t is small ($0 \leq t \leq 4$), the corn method 1 and the rice method 3 are used, the price vector with labor as numeraire being $p_I = (p_{corn} = p_{rice} = 1)$. For $t = 4$, the corn land 1 is fully cultivated. The next equilibrium ($4 \leq t \leq 4.5$), however, does not consist in extending the cultivation of corn to land 2 but in introducing cultivation of rice on land 4 jointly with land 3, while land 1 remains fully cultivated. This non-Ricardian equilibrium with two marginal rice methods is sustained by prices $p_{II} = (p_{corn} = 4, p_{rice} = 2)$: land 1 yields a positive rent and demand is met for activity levels $y_1 = \overline{\Lambda_1}, y_3(t) = 9 - 2t$ and $y_4(t) = 3t - 12$. For $t = 4.5$, land 3 is no longer cultivated and, for $t > 4.5$, a new Ricardian equilibrium is made of the marginal methods 2 and 4, land 1 being fully cultivated and land 3 being left fallow. An ultimate limit due to the scarcity of lands is eventually reached.

Some lessons are:

- In a non-Ricardian equilibrium, some good (corn in the above example) is produced only on fully cultivated lands and the methods which produce it do not matter for the determination of prices.

- The number of methods playing an active role in the determination of prices is however sufficient since some other industrial or agricultural good is produced by two methods simultaneously.

- Ricardo's general strategy is to discard rent from the analysis of value by considering the marginal methods only, because they pay no rent and determine the prices. The idea is to reduce the study of a system with scarce resources to that of a single-product system, in which each commodity is produced by a unique method. Long after Ricardo, such single-product systems have indeed been studied by Leontief and Sraffa and their formal properties elucidated by means of the Perron-Frobenius theorem. But that strategy fails because the standard properties of single-product systems do not hold: even if each method produces a unique commodity, the set M of marginal methods may contain two methods producing the same good and no method for another good and, therefore, is not a usual single-product system.

- When demand increases, the rice land 3 is first cultivated up to a certain maximum level reached for $t = 4$. Then, because of an external phenomenon (the corn land 1 is fully cultivated), the level of cultivation on land 3 decreases and drops to zero for $t = 4.5$. A comparison of these changes with those occurring on land 4, which is cultivated up from the level $t = 4$, shows that Ricardo's notion of 'order of cultivation', which presumes that a land cultivated for some level of demand remains cultivated for higher levels, does not apply.

- A number of conditions must be met for the occurrence of a non-Ricardian equilibrium. On the value side, the price vector sustained by both rice methods must be positive and such that some corn method yields a positive rent (otherwise, corn will not be produced at all). On the physical side, the activity levels of the second equilibrium describe the progressive transfer of rice from land 3 to land 4, which in the present case is an adequate answer to the demand for more corn because the rice method 4 is corn-saving with regard to the rice method 3.

3 Local and global dynamics

In this section, we look at the conditions for a smooth physical transition between consecutive equilibria. Due to the possible occurrence of non-Ricardian equilibria in the dynamics of the economy, there is no harm in assuming that the industrial or agricultural methods are of the multiple-product type, i.e. the productive method i ($i = 1, ..., m$) is represented by a input vector $a_i \in R^n_+$ of reproducible commodities, an amount $l_i \in R_+$ of homogenous labour and the area(s) $\Lambda_i \in R^l_+$ of land(s) used by the method at unit activity level (that unit is arbitrary, and $\Lambda_i = 0$ for industrial methods), while the output vector is a basket $b_i \in R^n_+$. We assume that all goods can be disposed of freely and that lands can be left fallow: fallowing is considered as a specific agricultural method with no inputs and outputs, the rent on the corresponding land being zero. (That formalization may look rather artificial, but it facilitates the counting of equations and the identification of the bounds to equilibrium, since a limit is then reached when some positive variable becomes zero.) At equilibrium, and flukes apart, the number of operated processes, including fallowing and free disposal, is equal to the number n of produced commodities plus the number l of lands: otherwise, either the degrees of freedom on the positive activity levels would not suffice to adapt the net product to the given demand basket, as described by the conditions

$$(B - A)y = d \tag{1}$$
$$\Lambda y = \overline{\Lambda} \tag{2}$$

or the prices-and-rents vector $(p, \rho) \in R^n_+ \times R^l_+$ would be overdetermined, since a price equality

$$(1 + r)p^T a_i + \rho^T \Lambda_i + l_i = p^T b_i \tag{3}$$

stating that the method yields the ruling rate of profits is associated with each operated process i. In this equality, following Sraffa (1960), we assume that the wage is paid *post factum*, but nothing is changed if it is paid at the beginning of the period. By setting $c_i(r) = (b_i - (1 + r)a_i, -\Lambda_i) \in R^{n+l}$ and $\pi = (p, \rho) \in R^{n+l}_+$, the same price equality is written more simply

$$\pi^T c_i(r) = l_i \tag{4}$$

Definition. *Given a set M of $n + l$ operated methods, $C_M(r)$ is the square matrix obtained by stacking the $n + l$ vectors $c_i(r)$ of the operated methods, and $C_M = C_M(0)$.*

With this notation, the price-and-wage vector is the solution of

$$\pi^T C_M(r) = l^T \tag{5}$$

Let us start from a given equilibrium for a certain demand $d = d(t)$ and change that demand slightly. In general, the activity levels $y(t)$ adapt themselves to the new demand vector. With the convention that fallowing is a method of production, lands are fully cultivated (possibly partly or totally with weeds) and the limit to the adaptation of activity levels is always reached when the activity level $y_j(t)$ of some method j, which was positive at $t = t_0 - \varepsilon$, drops to zero at $t = t_0$ and would become negative at $t_0 + \varepsilon$ (flukes apart, that method is unique). The existing equilibrium I thus fails for a physical reason, and we have to build the physical side and the value side of the next equilibrium II. The outgoing method j is well identified. In order to ensure a continuous physical transition between the previous and the next equilibrium II, the other $n + l - 1$ operated methods must continue to work at the same level, whereas some incoming method k is introduced at a zero level. How is that method selected? The point is that the choice of method k is uniquely determined by the value side of the problem: since the previous $n + l - 1$ marginal methods other than j satisfy condition (3) or (4) in both equilibria, the change $\Delta\pi = \pi_{II} - \pi_I$ in the price-and-rent vector satisfies the

$n+l-1$ linear equalities $\Delta\pi^T c_i(r) = 0$. The new price-and-rent vector π_{II} is therefore written as $\pi_{II} = \pi_I + \lambda\alpha$, where α is a given solution, unique up to a factor, of these equations, and λ is an unknown scalar. Moreover, the sign of λ is determined by the property that the outgoing method j pays extra-costs at the new price-and-rent vector: for a choice of α such that $\alpha^T c_j < 0$, λ is positive. On the whole, the incoming method is determined as follows: starting from $\lambda = 0$, let us increase λ and 'pivot' the prices $\pi_I + \lambda\alpha$ until some method k which was not operated becomes profitable. If the new price vector were not the one corresponding to the minimum value of λ satisfying that condition, method k would either yield extra-profits at the new equilibrium if it is an industrial method, or its activity level would jump from zero to maximum if it is an agricultural method. This is excluded in the Ricardian dynamics.

Note that equilibrium I and the candidate II have a specific relationship. All but one of their methods are identical and, by construction, the method j which is operated in I pays overcosts $\beta_j > 0$ at prices π_{II}, while the method k operated in II pays overcosts $\beta_k > 0$ at prices π_I. An algebraic expression of this property is obtained when, in the matrix $C_{II}(r)$, the incoming method k takes the column previously occupied by the outgoing method j. Then:

Lemma 1 *$\det C_I(r)$ and $\det C_{II}(r)$ have opposite signs.*

Proof. Since $\pi_{II}^T c_j(r) = l_j - \beta_j = \pi_I^T c_j(r) - \beta_j$ and $\pi_I^T c_k(r) = l_k - \beta_k = \pi_{II}^T c_k(r) - \beta_k$, we have $(\pi_{II} - \pi_I)^T(\beta_k c_j(r) + \beta_j c_k(r)) = 0$. As $(\pi_{II} - \pi_I)^T c_i(r) = 0$ for the other operated methods i, the matrix with the columns c_i and $\beta_k c_j + \beta_j c_k$ has a zero determinant. Therefore $\beta_k \det C_I(r) + \beta_j \det C_{II}(r) = 0$. ■

Clearly enough, as the unique candidate for the incoming method is selected on profitability considerations, that method does not necessarily meet the change in demand which caused the loss of the previous equilibrium (an exception is the golden rule hypothesis which ensures a strict duality between the value and the quantity sides). The condition for the local working of the Ricardian dynamics is that the new system is able to meet the demand vector $d(t_0 + \varepsilon)$, in which case the candidate II is indeed an equilibrium for that demand.

The matrix $C(r = 0)$ being denoted by C, the physical constraints (1)-(2) on activity levels are written as $Cy = (d(t), \overline{\Lambda})$. For $t = t_0 - \varepsilon$, all activity levels for equilibrium I are positive but the activity level $y_j(t)$ vanishes at $t = t_0$ and would become negative at $t = t_0 + \varepsilon$. When method j is replaced by another method, say method k, an algebraic decomposition of the vector

$d(t)$ leads to the formal equality

$$d(t) = \sum_{i \in M \setminus \{j\}} y_i(t)c_i + y_j(t)c_j = \sum_{i \in M \setminus \{j\}} y_i^{'}(t)c_i + y_k^{'}(t)c_k \tag{6}$$

At $t = t_0$ both decompositions coincide, with $y_i(t_0) = y_i^{'}(t_0) > 0$ and $y_j(t_0) = y_k^{'}(t_0) = 0$. At $t = t_0 + \varepsilon$, $y_i^{'}(t)$ remains positive by continuity and $y_k^{'}(t)$ has a small nonzero value. The new set of methods including method k sustains the demand vector $d(t_0 + \varepsilon)$ if and only if $y_k^{'}(t)$ is positive, whereas $y_j(t)$ has become negative.

Lemma 2 *The set of methods* $\{k\} \cup M \setminus \{j\}$ *can sustain the demand vector* $d(t_0 + \varepsilon)$ *if and only if* $\det C_I$ *and* $\det C_{II}$ *have opposite signs.*

Proof. Equality (6) shows that the $n + l - 1$ vectors c_i for $i \in M \setminus \{j\}$ and the vector $y_j(t)c_j - y_k^{'}(t)c_k$ are linearly dependent, therefore $y_j(t) \det C_I - y_k^{'}(t) \det C_{II} = 0$. $y_k^{'}(t)$ and $y_j(t)$ have opposite signs if and only if it is the same for the two determinants. ■

The condition on the sign of the two determinants can be considered as a productivity condition, that productivity being relative to the local variations of the demand vector. It follows from Lemmas 1 and 2 that:

Proposition. *Let the rate of profits be the given distribution variable. The Ricardian dynamics when demand changes work locally if and only if the sign of* $\det C_I(r)/\det C_I$ *is preserved when the outgoing method is replaced by the incoming method.*

As a consequence, the Ricardian dynamics always work when the rate of profit is zero and, by continuity, when it is small enough. They also hold in the case of single-product methods with extension of cultivation on specialized lands: simplifications then occur in the calculation of the determinants, and the signs of $\det C_I(r)$ and $\det C_I$ are deduced from those of $\det(I - (1+r)A)$ and $\det(I - A)$, which are both positive by the Hawkins-Simon criterion. On the contrary, the productivity condition matters in the case of pure intensive cultivation, as shown by Example 2 in the next Section.

If the condition stated in the Proposition holds for any pair of consecutive sets, the Ricardian dynamics work globally. Let us link a basket d_0 to another d_1 by a path d_t. An equilibrium at d_0 is progressively transferred, by means of successive transforms along the path, to another at d_1. If d_0 is sustained by a unique equilibrium, could another path joining d_0 to d_1 lead to a different equilibrium at d_1? Along the path, an equilibrium I is transformed into a uniquely defined equilibrium II when some activity level reaches its bound. The process is reversible in the sense that, if one starts from equilibrium II and moves in the opposite direction on the same path, its successor is equilibrium I (because, by construction, the equilibrium which succeeds equilibrium II is sustained by the price vector $\pi_{II} - \lambda\alpha$, which is equal to π_I). Imagine that demand d_1 is sustained by several equilibria. By following the path d_t, each of them is transferred to d_0 and two equilibria never merge during the successive transforms: otherwise, when moving in the opposite direction, an equilibrium would have two successors. As multiple equilibria at d_1 would give birth to multiple equilibria at d_0, the uniqueness property at d_0 ensures uniqueness for any demand basket.

4 Ricardo and Lemke

Lemke and Howson (1964) and Lemke (1965) introduced an algorithm to find a solution of a bimatrix game. The algorithm was soon adapted to a variety of problems, including linear and nonlinear complementarity problems (that class includes general equilibrium). The linear complementarity problem LCP(q, M), where q is a given vector in R^s and M an $s \times s$ matrix, consists in finding two nonnegative and orthogonal vectors z and w linked by the relationship $w = q + Mz$ (the orthogonality condition amounts to stating that w_i or z_i is zero for any component i). When free disposal and fallowing are not explicitly taken into account as methods of production, a long-term equilibrium can alternatively be defined as -a solution to a set of inequalities with complementarity relationships:

$$(B - A)y \leq d \quad [p] \tag{7}$$

$$\Lambda y \leq \overline{\Lambda} \quad [\rho] \tag{8}$$

$$p^T(B - (1 + r)A) - \rho^T\Lambda \leq l \quad [y] \tag{9}$$

The first inequality means that demand is met, the overproduced commodities having a zero price. The second inequality expresses the scarcity of lands, with a zero rent on non fully cultivated lands. The third inequality means that the operated methods yield the ruling rate of profits, and the non-operated methods do not yield more. With z obtained by stacking the price vector p, the rent vector ρ and the activity levels y, and q by stacking $d, \overline{\Lambda}$ and l, the system (7)-(8)-(9) is written as $w = Mz + q \geq 0$ where the vector w, obtained by stacking the excess supply, the areas of non cultivated lands and the overcosts, is complementary to z. An equilibrium is therefore a solution of the complementarity problem LCP(q, M) of dimension $s = n + l + m$. (Conversely, the equalities (1)-(2) are nothing but the inequalities (7)-(8) rewritten with slack variables.) The specific structure of matrix M is inessential for the present purpose.

The idea of the parametric LCP algorithm (Cottle *et al.*, 1993) is to find a solution to LCP(q_1, M) by transferring a given solution to LCP(q_0, M) along the path $q(t) = q_t = (1-t)q_0 + tq_1$. A solution at q_0, more generally at q_t, is characterized by a set $\mathcal{S}_0$ of s zero components of vector (z, w), the other s components being strictly positive in normal times. Slight changes in q_t are met by adjusting these positive components. A limit is reached when some positive component, say $w_j(t)$, vanishes at $t = t_0$. Then the next set of solutions is obtained by solving the affine system $w = Mz + q$ and imposing the same zero components to (z, w), except that the constraint $w_j = 0$ is substituted for the previous one $z_j = 0$. In a neighborhood of t_0, the $s-1$ positive components of $(z(t_0), w(t_0))$ remain positive by continuity whereas $z_j(t)$, which is zero for $t = t_0$, is close to zero at $t_0 + \varepsilon$.

It turns out that the parametric LCP algorithm is nothing but a version of the Ricardian dynamical model in which vector q plays the role of an augmented demand vector. In general, the transfer from q_t to $q_{t+\varepsilon}$ sets no problem, with a rupture from time to time when a positive variable drops to zero. Then the new system is a neighbor of the previous, in the sense that only one condition is changed. The search for an incoming method as described in Section 3 is also similar to that of a new facet in the LCP problem.

A significant difference, however, occurs when $z_j(t_0+\varepsilon)$, which we know to be close to zero, is negative. Then, the Ricardian dynamics stop because the incoming method is not productive enough. By contrast, the Lemke algorithm goes on by making a U-turn on the path: since $z_j(t_0 - \varepsilon)$ is then positive, a second solution to LCP(q_t, M) for $t = t_0 - \varepsilon$ is found. (The relative sign of two determinants is reversed with regard to that of the initial sequence of equilibria, and this sign plays a role similar to that of the index in the differentiable version of general equilibrium theory (Mas Colell, 1990).) The algorithm continues for lower and lower values of t ('antitone' move, in the terminology of the LCP literature). Eventually, however, another reverse may occur (or must occur, if the conditions ensuring the working of the parametric algorithm are met) and after the change of several methods of production, a new long-term equilibrium sustains increasing levels of demand. A third solution sustaining $q(t_0 - \varepsilon)$ is first reached and, eventually, a solution for level q_1, as an example shows:

Example 2. Let corn be the unique good and consider three methods of cultivation on a homogenous land with total area $\overline{\Lambda} = 20$ acres. The methods are:

method 1 : 3 corn + 4 labor + 4 acres $\rightarrow$ 10 corn

method 2: 2 corn + 7 labor + 5 acres → 10 corn
method 3: 3 corn + 21 labor + 1 acre → 10 corn

The given rate of profit is $r = 1$. For low levels of demand, method 1 is the cheapest (the price of corn in terms of wage is $p_1 = 1$, and the rent ρ_1 is zero). At level $d = 35$, the land is fully cultivated and the Ricardian dynamics stop because the next cheapest method is method 2, which is less productive per acre than method 1. On the contrary, the Lemke algorithm goes on: at price $p_{1,2} = 2$ and rent $\rho_{1,2} = 1$, method 2 is progressively substituted for method 1 and the net product drops to 32. Eventually, at price $p_{2,3} = 7$ and rent $\rho_{2,3} = 7$, method 3 is introduced jointly with method 2 and, after a progressive substitution, the net product increases up to 140.

Two comments are in order. First, the phenomenon we point at remains if the number of methods was a continuum. Second, the solutions to Example 2 give the set of equilibria for any level of demand, because there is only one good. In a more general framework, the powerfulness of the Lemke algorithm is illustrated by the following observation. Assume that d_0 is sustained by a unique equilibrium (which is the case for single-product systems if d_0 is small), and the same for d_1 (for instance, d_1 is high). For another demand vector d, let us draw a path from d_0 to d_1 with $d_{0.5} = d$. Crucial properties of the Lemke algorithm are that the sequence of equilibria generated along the path is made of distinct techniques (the algorithm does not cycle) and that the passage from a technique to the next is uniquely determined and reversible. The algorithm which starts from the unique equilibrium at d_0 will eventually reach the unique equilibrium at d_1. The algorithm which starts from a given equilibrium sustaining d will not stop before reaching the unique equilibrium sustaining either d_0 or d_1. Therefore the second path is a part of the first. This shows that *any* equilibrium at d is reached on the path d_t (exhaustivity property).

5 References to the literature

The revival of analytical Ricardian studies is due to Sraffa (1960). Sraffa rejected Ricardo's idea of a natural order of cultivation because the decision to cultivate some land rather than another depends on the relative costs, which are sensitive to the distribution of income. Montani (1975) showed that, for a given rate of profits and in the case of pure extensive rent, the order of cultivation coincides with that of the decreasing wages paid by the agricultural methods for a zero rent. By contrast, the Ricardian dynamics do not work automatically in the case of pure intensive rent (Bidard, 2010).

Probably under the influence of Sraffa's preface which stresses that no change in quantities is ever considered (though Chapter 11 on lands makes a clear reference to the Ricardian dynamics), most post-Sraffian studies assume a given level of demand. From this abundant and intricate literature, we retain for our purpose the recognition of multiple equilibria (D'Agata, 1982; Freni, 1991) and of non-Ricardian equilibria (Erreygers 1990), a general existence result based on the identification with a linear complementarity problem (Salvadori, 1986) and the studies on uniqueness (Erreygers 1990, 1995): starting from geometrical considerations, Erreygers elaborates a criterion for local uniqueness identical to that mentioned in the Proposition and states a global uniqueness result (which, in our views, is not totally convincing).

6 Conclusion

Ricardo's idea was to get rid of lands and rents by reducing the core of the productive system to the marginal methods. The existence of non-Ricardian equilibria, for which some good is not produced by a marginal method, shows the limit of the approach. But the dynamics can only be facilitated by the presence of such equilibria, since the introduction of a corn-saving method in some industry may be an adequate answer to a global lack of corn. Even then, the working of the Ricardian dynamics comes up against a productivity condition: when some long-term equilibrium fails because it cannot meet the increase of demand, the reconstruction of another equilibrium starts from profitability considerations, so that the incoming method may be unable to follow the evolution of demand. That productivity condition is expressed as a simple algebraic criterion.

The lessons of the study go well beyond the case of lands: little would be changed if the scarcity constraints concerned consumption or production goods. A formal reason for that generality is that the Ricardian dynamics turn out to be very close to an algorithm used to solve a class of problems often encountered in economics. The main difference is that the Ricardian dynamics, which allow for discontinuities in prices and rents, pays attention to the smoothness of the physical transition between equilibria, and there is no doubt that this condition is economically sensible. The domain of validity of the Ricardian dynamics can be enlarged by taking into account the influence of relative prices on demand: allowing for substitution in final demand alleviates the physical constraints on the scarce goods and facilitates the working of the dynamics.

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