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Complementarity Problems and General Equilibrium

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Abstract. A general equilibrium technique is used to show the existence of a solution to a nonlinear complementarity problem involving a copositive function.

Keywords. Nonlinear complementarity problem, general equilibrium, copositivity.

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Research on the bimatrix game is at the origin of algorithmic methods [6, 7] which are now widely used to find the solutions of linear complementarity problems (LCP). The note proposes an existence proof for nonlinear complementarity problems (NCP): given a continuous function $w : \mathbf{z} \in R^n_+ \rightarrow \mathbf{w} = w(\mathbf{z}) \in R^n$, the problem consists of finding a pair $(\mathbf{w}, \mathbf{z})$ of nonnegative and orthogonal vectors. The proof is non algorithmic but relies on the search of a fixed point which can be implemented by an algorithm.

Most algorithmic methods start from nonnegative variables and proceed to make them orthogonal. One may imagine an alternative approach which starts from orthogonal variables and adjusts them in order that they become nonnegative. The problem is then similar to that of the existence of a general equilibrium in economics: the excess supply function is always orthogonal to the price vector (Walras identity) and an equilibrium is reached when the excess supply is nonnegative. Since the existence result follows from the Gale-Nikaido-Debreu Lemma [4, 5, 8], the parallel suggests that it is possible to adapt it to the NCP.

Notation $\mathbf{x} \geq \mathbf{0}$ means that the column vector $\mathbf{x}$ is nonnegative, $\mathbf{x} > \mathbf{0}$ that it is nonnegative and nonzero, and $\mathbf{x}'$ denotes its transpose. Notation $\|\mathbf{x}\|$ designates the norm $\|\mathbf{x}\| = \sum_i |x_i|$. The unit simplex of R^{n+1}_+ is denoted by S.

Lemma 1 *(GND Lemma). Let C be a compact and convex set in R^n and $s : \mathbf{z} \in C \rightarrow \mathbf{s} = s(\mathbf{z}) \in R^n$ a continuous function satisfying the Walras inequality: $\forall \mathbf{z} \in C \;\; \mathbf{z}' s(\mathbf{z}) \geq 0$. There exists $\overline{\mathbf{z}} \in C$ such that $\overline{\mathbf{s}} = s(\overline{\mathbf{z}})$ belongs to the polar cone of C, i.e. $\forall \mathbf{z} \in C \;\; \mathbf{z}' \overline{\mathbf{s}} \geq 0$.*

Proof. The image of the set C by s is included in a convex compact set K. Let us consider the point-to-set mapping φ from K to C defined by $\varphi(\mathbf{x}) = \left\{ \mathbf{z}; \mathbf{z} \in C \;\; \mathbf{z}'\mathbf{x} = \min_{\mathbf{y} \in C} \mathbf{y}'\mathbf{x} \right\}$. The correspondence $s \times \varphi$ from $C \times K$ into $K \times C$ being upper hemicontinuous with compact convex values, the existence of a fixed point $(\overline{\mathbf{z}}, \overline{\mathbf{s}})$ follows from the Kakutani theorem. Then $\min_{\mathbf{y} \in C} \mathbf{y}' \overline{\mathbf{s}} = \varphi'(\overline{\mathbf{s}}) \overline{\mathbf{s}} = \overline{\mathbf{z}}' s(\overline{\mathbf{z}}) \geq 0$. ■

The usual version of the Lemma assumes that C is the unit simplex of R^n, in which case the conclusion is that $\overline{\mathbf{s}}$ is nonnegative. Starting from a nonlinear function $w(\mathbf{z})$, we introduce an additional scalar t in order to obtain a homogenous function and enforce the Walras identity. It is then possible to apply the Lemma, a method often used in [1] to study prices of production. The basic hypothesis we retain on w is:

(H): The function $\overline{w}(\mathbf{z},t) = tw(\frac{\mathbf{z}}{t})$ defined on the set $\{\mathbf{z} \geq \mathbf{0}, t > 0\}$ admits a continuous extension to the set $\{\mathbf{z} \geq \mathbf{0}, t = 0\}$.

Since $\overline{w}$ is homogenous of degree one, so is its extension $\overline{w}(\mathbf{z},0)$ which is denoted by $m(\mathbf{z})$. Therefore hypothesis (H) amounts to assuming that function w admits a (unique) decomposition $w = m + q$, where m is homogenous and q is such that $\lim\limits_{t\to 0} tq(\frac{\mathbf{z}}{t}) = 0$. The corresponding NCP is a generalization of the linear complementarity problem LCP($\mathbf{q},\mathbf{M}$), for which $q(\mathbf{z}) = \mathbf{q}$ and $m(\mathbf{z}) = \mathbf{Mz}$.

Theorem 1 *Let $w : R^n_+ \to R^n$ such that $w = m + q$, m being continuous and homogenous of degree one, and q continuous and such that $\lim\limits_{t\to 0} tq(\frac{\mathbf{z}}{t}) = 0$. Let $E = \{\mathbf{z} \in S, m(\mathbf{z}) \geq \mathbf{0}, \mathbf{z}'m(\mathbf{z}) = 0\}$. If the following properties hold:*

$$\liminf_{\mathbf{z}\in S, t<1} \mathbf{z}'w(\frac{\mathbf{z}}{t}) \quad > \quad -\infty \tag{1}$$

$$\mathbf{z} \quad \in \quad E \Rightarrow \liminf_{\mathbf{z}_t\to\mathbf{z}, t<1} \mathbf{z}_t'w(\frac{\mathbf{z}_t}{t}) > 0 \tag{2}$$

there exists a solution to the complementarity problem

$$w(\mathbf{z}) \geq \mathbf{0} \qquad [\mathbf{z}] \tag{3}$$

Proof. Let S_ε be the subset of the unit simplex S of R^{n+1}_+ made of the vectors $(\mathbf{z},t)$ with $t \geq \varepsilon > 0$. Since the continuous function defined on S_ε by $s(\mathbf{z},t) = (tw(\frac{\mathbf{z}}{t}), -\mathbf{z}'w(\frac{\mathbf{z}}{t}))$ satisfies the Walras identity $(\mathbf{z},t)'s(\mathbf{z},t) = 0$, the GND Lemma asserts the existence of a point $(\mathbf{z}_\varepsilon, t_\varepsilon) \in S_\varepsilon$ such that:

$$\forall(\mathbf{z},t) \in S_\varepsilon \quad \mathbf{z}'(t_\varepsilon w(\frac{\mathbf{z}_\varepsilon}{t_\varepsilon})) + t(-\mathbf{z}_\varepsilon' w(\frac{\mathbf{z}_\varepsilon}{t_\varepsilon})) \geq 0 \tag{4}$$

Assume first that the sequence $(\mathbf{z}_\varepsilon, t_\varepsilon)$ admits a cluster point $(\mathbf{z}_*, t_*) \in S$ with $t_* > 0$. As any given point $(\mathbf{z},t) \in S$ with $t > 0$ belongs to S_ε for ε small enough, inequality (4) still holds at the limit: $\mathbf{z}'(t_* w(\frac{\mathbf{z}_*}{t_*})) + t(-\mathbf{z}_*' w(\frac{\mathbf{z}_*}{t_*})) \geq 0$. It also holds by continuity for any $(\mathbf{z},t) \in S$. Therefore $t_* w(\frac{\mathbf{z}_*}{t_*}) \geq 0$ and $-\mathbf{z}_*' w(\frac{\mathbf{z}_*}{t_*}) \geq 0$, so that $\frac{\mathbf{z}_*}{t_*}$ is a solution to (3).

Assume now that $\lim t_\varepsilon = 0$ (hence $\mathbf{z}_* \in S$). Since condition (1) amounts to stating the existence of a scalar α such that $\mathbf{z}'w(\frac{\mathbf{z}}{t}) > -\alpha \|\mathbf{z}\|$ for any $\|\mathbf{z}\| \geq 0.5$ and $0 < t < 1$, we have $-\mathbf{z}_\varepsilon' w(\frac{\mathbf{z}_\varepsilon}{t_\varepsilon}) < \alpha \|\mathbf{z}_\varepsilon\| < \alpha$. And since inequality (4) applied at point ($\mathbf{z} = \mathbf{0}$, $t = 1$) shows that $-\mathbf{z}_\varepsilon' w(\frac{\mathbf{z}_\varepsilon}{t_\varepsilon}) \geq 0$, there exists a sequence ε such that $\lim -\mathbf{z}_\varepsilon' w(\frac{\mathbf{z}_\varepsilon}{t_\varepsilon}) = l \geq 0$. It follows again from (4) that inequality $\mathbf{z}'m(\mathbf{z}_*) + tl \geq 0$ holds for any point $(\mathbf{z},t) \in S$ with $t > 0$, therefore $m(\mathbf{z}_*) \geq \mathbf{0}$. Moreover, we have $\mathbf{z}_*'m(\mathbf{z}_*) = \lim \mathbf{z}_\varepsilon' t_\varepsilon w(\frac{\mathbf{z}_\varepsilon}{t_\varepsilon}) =$

$(\lim t_\varepsilon)(\lim \mathbf{z}_\varepsilon' w(\frac{\mathbf{z}_\varepsilon}{t_\varepsilon})) = 0(-l) = 0$. To sum up, $\mathbf{z}_*$ belongs to the set E and the sequence $(\mathbf{z}_\varepsilon, t_\varepsilon)$ violates condition (2). This case is therefore excluded. ■

Given $\mathbf{z} \in S$, condition (1) implies that $\mathbf{z}' m(\frac{\mathbf{z}}{t}) + \mathbf{z}' q(\frac{\mathbf{z}}{t}) > -\alpha$, therefore $\mathbf{z}' m(\mathbf{z}) > -\alpha t - \mathbf{z}'\ t q(\frac{\mathbf{z}}{t})$ for any $t \in]0, 1]$. Letting t tend to zero shows that $\mathbf{z}' m(\mathbf{z})$ is nonnegative (i.e., function m is copositive). The conditions (1) and (2) deal with the behavior of function w at infinity, in which sense a parallel can be made with the coercivity hypotheses often referred to in the theory of nonlinear complementarity and variational inequalities (e.g., [3]). However, they do not require that w 'grows' at infinity.

We consider in particular the problem NCP$(\mathbf{q}, f)$

$$f(\mathbf{z}) + \mathbf{q} \geq \mathbf{0} \qquad [\mathbf{z}] \tag{5}$$

associated with the function w defined by $w(\mathbf{z}) = f(\mathbf{z}) + \mathbf{q}$, where f is a given continuous function and $\mathbf{q}$ is a given vector of R^n.

Corollary 1 *Let $f : R^n_+ \to R^n$ be continuous and let function $\overline{f}(\mathbf{z}, t) = t f(\frac{\mathbf{z}}{t})$ admit a continuous extension when t tends to zero, with $m(\mathbf{z}) = \overline{f}(\mathbf{z}, 0)$. If $\liminf_{\|z\| \to \infty} \mathbf{z}' f(\mathbf{z}) > -\infty$, NCP$(\mathbf{q}, f)$ admits a solution for any vector $\mathbf{q}$ such that*

$$\mathbf{z} \in E \Rightarrow \mathbf{z}'\mathbf{q} > 0 \tag{6}$$

Proof. As $\overline{w}(\mathbf{z}, t) = \overline{f}(\mathbf{z}, t) + t\mathbf{q}$, the extension of $\overline{w}$ at $t = 0$ coincides with that of $\overline{f}$. For $\mathbf{z} \in S$, $\mathbf{z}' w(\frac{\mathbf{z}}{t}) = t \frac{\mathbf{z}'}{t} f(\frac{\mathbf{z}}{t}) + \mathbf{z}'\mathbf{q} > -\alpha t + \mathbf{z}'\mathbf{q}$, therefore $\liminf_{\mathbf{z} \in S, t<1} \mathbf{z}' w(\frac{\mathbf{z}}{t}) \geq \min_{\mathbf{z} \in S} \mathbf{z}'\mathbf{q} > -\infty$ and $\liminf_{\mathbf{z} \in E, t<1} \mathbf{z}' w(\frac{\mathbf{z}}{t}) \geq \min_{\mathbf{z} \in E} \mathbf{z}'\mathbf{q} > 0$, so that conditions (1) and (2) are both met and the existence theorem applies. ■

An extension of the above statement and argument to the case when vector $\mathbf{q}$ is replaced by a continuous function $q(\mathbf{z})$ with bounded values is immediate.

The corollary can be compared with a well known result concerning the linear complementarity problem LCP$(\mathbf{q}, \mathbf{M})$ when $\mathbf{M}$ is a copositive matrix. That result ensures the existence of a solution under the weaker condition that $\mathbf{z} \in E$ implies $\mathbf{z}'\mathbf{q} \geq 0$ (Theorem 3.8.6 in [2]). In the nonlinear case, however, some restriction is unavoidable, as shown by a numerical example: consider the problem NCP$(\mathbf{q}, f)$ for $n = 2$, $\mathbf{q}' = (0, -1)$ and f defined by $f(\mathbf{0}) = \mathbf{0}$ and $f(z_1, z_2) = \frac{z_2}{z_1 + z_2}\ (-z_2, z_1)$ for $\mathbf{z} > \mathbf{0}$. f is copositive and homogenous of degree one, therefore its extension m coincides with f itself. As $\{\mathbf{z} \geq \mathbf{0}, f(\mathbf{z}) \geq \mathbf{0}\}$ requires $z_2 = 0$, the condition $\mathbf{z} \in E$ does imply

$\mathbf{z}'\mathbf{q} \geq 0$. However, NCP$(\mathbf{q}, f)$ has no solution since there is no $\mathbf{z} \geq \mathbf{0}$ such that $f(\mathbf{z}) + \mathbf{q} \geq \mathbf{0}$. Similarly, hypothesis $\liminf_{\|z\| \to \infty} \mathbf{z}' f(\mathbf{z}) > -\infty$ in the corollary, which is clearly weaker than the condition 'f is copositive', cannot be replaced by the still weaker condition 'm is copositive', as shown by another example: $\mathbf{q}' = (1, 1)$ and $f(z_1, z_2) = (\frac{1}{z_1+1}, -2)$.

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