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Robustness of equilibrium price dispersion in finite market games

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Abstract

We propose an approach to restricting the set of equilibria in a strategic market game and use it to assess the robustness of the price dispersion results obtained by Koutsougeras [2003, J. Econ. Theory 108, 169–175] in the multiple trading posts setup. More precisely, we perturb the initial game by the introduction of transaction costs and our main results are the following. (i) No equilibrium with price dispersion of the game with costless transactions can be approached by equilibria with positive transaction costs as costs get arbitrarily small. (ii) When this type of perturbation is considered the set of equilibrium outcomes is not affected by the number of trading posts.

Keywords: Strategic market games, law of one price, equilibrium selection.

JEL Classification: C72, D43, D50.

Resumé

Nous proposons une approche permettant de réduire l'ensemble des équilibres dans un jeu stratégique de marché, et l'appliquons pour étudier la robustesse du résultat de non uniformité des prix obtenu par Koutsougeras [2003, J. Econ. Theory 108, 169–175] dans le jeu de marché à postes d'échanges multiples. Plus précisément, nous perturbons le modèle initial par l'introduction de coûts de transaction positifs et nos résultats principaux sont les suivants. (i) Aucun équilibre avec dispersion de prix du jeu sans coûts de transaction ne peut être approché par des équilibres avec coûts (strictement) positifs tendant vers zéro. (ii) Lorsque ce type de perturbations est considéré, l'ensemble des allocations d'équilibres est invariant avec le nombre de postes d'échanges.

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1 Introduction

In any economic model, the assumption that trade is costless is to be considered as a simplification. One generally expects that the model's predictions hold approximately for small but positive transaction costs. Formally, the approximation is warranted whenever there is some form of continuity as transaction costs vanish. In contrast, a property that does not hold—even approximately—once arbitrarily small transaction costs are explicitly modeled may be viewed as artificial.

In this paper, we apply this argument to qualify recent results about equilibrium price dispersion in strategic market games. To be precise, we consider the multiple trading posts per commodity variant of two canonical market games (see below). As shown by Koutsougeras (1999, 2003*b*) this framework exhibits—along with equilibria with uniform prices—equilibria where prices are *not equalized* among posts where the same good is traded. Hence, the 'law of one price' may fail in an exchange economy with costless trade. [1]

We show that this striking result as well as related properties of the multiple posts setup are not immune to the introduction of arbitrarily small transaction costs. More precisely we perturb the initial game by the introduction of (a simple form of) transaction costs and we obtain two sets of results. Regarding price dispersion, we first show that when transaction costs are positive, any equilibrium must satisfy the law of one price. Our main result then states that an equilibrium with price dispersion of the game with costless transactions cannot be approached by equilibria with positive transaction costs as costs get arbitrarily small—in short, we say that such an equilibrium is not "robust".

We also show that the set of robust outcomes—viz, those outcomes associated with robust equilibria—does not depend on the number of trading posts. This investigation is motivated by the analysis in Koutsougeras (2003*a*) indicating that the emergence of equilibria with dispersed price is the only difference between the single and the multiple posts variants. In contrast, our irrelevance result suggests that there is no loss of generality in working with the canonical, single trading post market game.

The intuition for why (small) transaction costs restaure the law of one price is as follows. As shown by Gobillard (2006), [2] the result of Koutsougeras (2003*b*) relies on agents placing 'wash-sales' trades, that is bids and offers that cancel each other *on a same trading post*. Although wash-sales trades matter for the strategic equilibria because they affect the relative thickness of trading posts, any agent is indifferent as to the amount of his own wash-sales (see Postlewaite and Schmeidler, 1978 and Peck, Shell and Spear, 1992). In other words, an agent's allocation only depends on his net trades. With strictly

[1] And, importantly, in an environment in which agents face no liquidity constraint in their arbitrage strategy (Koutsougeras, 2003*b*).

[2] See also proposition 9 in Bloch and Ferrer (2001).

positive transaction costs, agents also care about their gross trades, and never choose to be active on both sides of a given post (lemma 1). In the limit where transaction costs vanish, this leads to the selection—among the best response strategies—of the unique strategy minimizing gross trade, that is that without wash-sales. One consequence is that price dispersion is no longer compatible with equilibrium.

We derive our results for two contrasting frameworks: the multiple trading posts extension of the inside money market game of Postlewaite and Schmeidler (1978), and that of the commodity money market game à la Shapley and Shubik (1977) and Dubey and Shubik (1978). Examples of equilibria where the law of one price fails are given by Koutsougeras (1999) for the latter setup, and by Koutsougeras (2003a,b) for the former. The key difference between both setups is that in the commodity money case agents' trading strategies are constrained by their money holding. Our result shows that liquidity constraints *per se* do not induce price dispersion.

Our specification borrows from Rogawski and Shubik (1986), who introduce transaction costs paid in commodities in the bid-offer market game of Dubey and Shubik (1978). Their main concern is the existence of an equilibrium with active trade when transaction costs are not too prohibitive. Instead, we use transaction costs to define perturbed games and to reduce the set of equilibria. We choose the simplest specification—homogeneous linear costs—necessary to make our point.

From a more general perspective, the paper proposes a natural approach to restricting the set of equilibria in a market game. By eliminating the indeterminacy of best responses associated with wash-sales, our argument cuts down one source for the multiplicity of (Nash) equilibria in market games (Peck et al., 1992).[3] So far, equilibrium selection in market games has centered around a distinct issue, that is that any market can be active or inactive in equilibrium. In particular, the trivial Nash equilibrium in which all bids and offers are zero always exists (Shapley, 1976). Following Dubey and Shubik (1978), one popular approach has been to consider equilibria obtained as the limit of perturbed games in which some outside agency places vanishingly small bids and offers on each trading post.[4] More recently, Matros and Temzelides (2004) use some strong notion of evolutionary stability to rule out equilibria in which (some) markets are inactive.

The paper is organized as follows. In section 2 we introduce the framework and some definitions. We prove the results for the market game with inside money in section 3. In section 4 we show that our results extend to the market game with commodity money. Some general comments are drawn in section

[3] In a very different context Peck (2003) shows that the multiplicity of best responses is not robust to the introduction of arbitrarily small demand uncertainty.
[4] It is worth noticing that the selection of a non trivial Nash equilibrium is not guaranteed as shown by Cordella and Gabszewicz (1998) and Busetto and Codognato (2004).

5. One proof is relegated to an appendix.

2 General setting and definitions

We consider an exchange economy with a finite set $\mathcal{H}$ of agents, indexed by $h = 1, ..., H$, and $L+1$ goods, indexed by $i = 1, ..., L+1$. Commodity $L+1$ represents money. Each agent $h \in \mathcal{H}$ has endowment $e_h \in I\!R_+^{L+1}$, and preferences described by a utility function $u_h : I\!R_+^{L+1} \longrightarrow I\!R_+$ defined over consumption bundles. We require u_h to be continuous, strictly increasing in the consumption of goods $1, \ldots, L$ and non decreasing in money.

Trade is organized as follows. For any commodity $i = 1, ..., L$, there are $K^i \geq 1$ trading posts where good i is exchanged for money (good $L+1$). Trading posts are indexed by (i, s), where $s = 1, ..., K^i$. We let $K = \sum_{i=1}^{L} K^i$ denote the aggregate number of trading posts in the economy.

An agent's strategy, σ_h, specifies for each trading post a non negative offer of commodity, $q_h^{i,s}$, and a non negative bid in term of money, $b_h^{i,s}$. Let S_h denote the strategy set of agent h, and $S = S_1 \times \ldots \times S_H$ the set of strategy profiles, with generic element $\sigma = (\sigma_h)_{h \in \mathcal{H}}$. To single out the strategy of a given agent h, we will sometimes write the strategy profile as (σ_h, σ_{-h}).

Given a strategy profile $\sigma \in S$, define

$$B^{i,s} = \sum_{\mathcal{H}} b_h^{i,s} \qquad \text{and} \qquad Q^{i,s} = \sum_{\mathcal{H}} q_h^{i,s},$$

and, for a given $h \in \mathcal{H}$,

$$B_h^{i,s} = \sum_{\mathcal{H} \setminus \{h\}} b_{h'}^{i,s} \qquad \text{and} \qquad Q_h^{i,s} = \sum_{\mathcal{H} \setminus \{h\}} q_{h'}^{i,s}.$$

On trading post (i, s), prices are formed according to the standard Shapley-Shubik rule:

$$p^{i,s} = \begin{cases} B^{i,s}/Q^{i,s} & \text{if } Q^{i,s} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Subsequently we use the convention $1/p^{i,s} = 0$ whenever $p^{i,s} = 0$.

We follow Rogawski and Shubik (1986) in modeling transaction costs as consuming part of the commodities offered in transaction. We further restrict ourselves to the following linear and homogeneous specification. When an agent offers a quantity $q_h^{i,s}$ of commodity i on post (i, s), an additional quantity $\varepsilon q_h^{i,s}$ ($\varepsilon \geq 0$) is needed in order to place that offer. [5] The assumption that transaction costs do not depend on bids $b_h^{i,s}$ is a simplification.

[5] That is, some proportional fraction of goods disappears in transaction. This corresponds to an "iceberg cost" specification.

The description of the mapping from strategy profiles to final allocations is deterred to sections 3 and 4 because the details slightly differ between both settings. For $\varepsilon = 0$, our market games reduce to those analyzed in Koutsougeras (2003*b*) and Koutsougeras (1999). We will refer to the initial market game with inside money (respectively commodity money) as Γ (resp. Γ') and to the game with a given $\varepsilon > 0$ as Γ_ε (resp. Γ'_ε).

The rest of this section is devoted to the formal statement of the law of one price, and to our robustness requirement.

Definition 1 *A trading post* (i, s) *is active if* $p^{i,s} > 0$, *or equivalently* $B^{i,s} > 0$ *and* $Q^{i,s} > 0$.

Definition 2 *A strategy profile* $\sigma \in S$ *satisfies the law of one price if it induces—for any good* $i = 1, \ldots, L$*—prices that are uniform across active trading-posts:*

$$\left(p^{i,s}p^{i,r} > 0 \Longrightarrow p^{i,s} = p^{i,r}\right) \qquad \forall\; i, r, s.$$

We say that a Nash equilibrium of the initial game is robust if it can be approached by equilibria of the perturbed games with strictly positive transaction costs as transaction costs vanish. Formally,

Definition 3 *A Nash equilibrium* σ *of the game* Γ *is "robust" if there exists a sequence* $\{^n\varepsilon, ^n\sigma\}_{n=1}^{\infty}$ *where* $^n\varepsilon \in I\!R_+$ *and* $^n\sigma\,(\in S)$ *is a NE of the perturbed games* $\Gamma_{^n\varepsilon}$ *such that* $\lim_{n\to\infty} {^n\varepsilon} = 0$ *and* $\lim_{n\to\infty} {^n\sigma} = \sigma$.

3 The market game with inside money

In this section, we derive our results for the market game with inside money introduced by Postlewaite and Schmeidler (1978), and extended to multiple trading posts by Koutsougeras (2003*b*).

Good $L+1$ is an inside money with no direct utility. Agents have no initial money endowment, $e_h^{L+1} = 0$, and can issue inside money at no cost. To avoid over issuance, it is postulated that an agent that goes bankrupt—that is whose (monetary) gains from sales do not cover his bids—has all his bids and offers confiscated (see Peck et al. (1992) for a discussion).

Formally, agent h chooses his strategy σ_h in the set [6]

$$S_h = \left\{ \left(b_h^{i,s}, q_h^{i,s}\right) \in I\!R_+^{2K} \quad | \quad \sum_{s=1}^{K_i} (1+\varepsilon)\, q_h^{i,s} \le e_h^i \right\},$$

[6] To ease the exposition, we do not write S_h^ε for the strategy set although it does depend on ε. Note that strategy sets for any $\varepsilon > 0$ are included into strategy sets for $\varepsilon = 0$, so that any equilibria belongs to this larger set, $S_1^0 \times \cdots \times S_H^0$. No confusion should result.

and, given others' strategies, does not go bankrupt whenever

$$D_h(\sigma_h, \sigma_{-h}) := \sum_{i=1}^{L}\sum_{s=1}^{K_i} b_h^{i,s} - \sum_{i=1}^{L}\sum_{s=1}^{K_i} q_h^{i,s} p^{i,s} \leq 0. \tag{2}$$

Final allocations are then determined as follows:

$$x_h^i(\sigma) = \begin{cases} e_h^i - (1+\varepsilon)\sum_{s=1}^{K_i} q_h^{i,s} + \sum_{s=1}^{K_i} b_h^{i,s}/p^{i,s} & \text{if (2) holds,} \\ e_h^i - (1+\varepsilon)\sum_{s=1}^{K_i} q_h^{i,s} & \text{otherwise.} \end{cases} \tag{3}$$

It easily follows from (3) that (2) holds with equality at the optimum for h.

With a slight abuse in notation, a Nash equilibrium for Γ_ε is a strategy profile $\sigma \in S$ such that

$$\forall h \in \mathcal{H} \quad u_h(\sigma_h, \sigma_{-h}) = \sup_{\hat{\sigma}_h \in S_h} u_h(\hat{\sigma}_h, \sigma_{-h}).$$

3.1 Intermediate results

We start with one intermediate result stating that when $\varepsilon > 0$ it is never a best reply to buy and sell on the same trading post.

Lemma 1 *Let $\varepsilon > 0$. Any individual best reply in Γ_ε satisfies $b_h^{i,s} \cdot q_h^{i,s} = 0$.*

PROOF. Assume the contrary, viz $b_h^{i,s} > 0$ and $q_h^{i,s} > 0$ for a candidate best reply σ_h. We construct a profitable deviation $\hat{\sigma}_h$ by subtracting a small amount of wash-sales (conveniently defined) on post (i,s). Formally, for $\eta > 0$ consider the deviation with $\hat{b}_h^{i,s} = b_h^{i,s} - \eta$ and $\hat{q}_h^{i,s}$ defined by

$$-\hat{q}_h^{i,s} + \hat{b}_h^{i,s}\frac{Q_h^{i,s} + \hat{q}_h^{i,s}}{B_h^{i,s} + \hat{b}_h^{i,s}} = -q_h^{i,s} + b_h^{i,s}\frac{Q_h^{i,s} + q_h^{i,s}}{B_h^{i,s} + b_h^{i,s}}. \tag{4}$$

For $\eta > 0$ small enough, this deviation is admissible. Indeed, $b_h^{i,s} > 0$ and $q_h^{i,s} > 0$ imply that we can choose $\eta > 0$ such that $\hat{b}_h^{i,s} > 0$ and $\hat{q}_h^{i,s} > 0$. Noting that $\hat{q}_h^{i,s} < q_h^{i,s}$ from (4), we have $\hat{\sigma}_h \in S_h$. Now $\hat{\sigma}_h$ satisfies condition (2) because substituting $\hat{\sigma}_h$ for σ_h affects neither prices nor the (net) commodity bundle obtained through trading. [7] Formally, straightforward manipulations show that $D_h(\hat{\sigma}_h, \sigma_{-h}) = D_h(\sigma_h, \sigma_{-h})$. Furthermore, by playing $\hat{\sigma}_h$, agent h gets the same allocation $x_h^j(\hat{\sigma}_h, \sigma_{-h}) = x_h^j(\sigma_h, \sigma_{-h})$ for all goods $j \neq i$, and $x_h^i(\hat{\sigma}_h, \sigma_{-h}) - x_h^i(\sigma_h, \sigma_{-h}) = \varepsilon\left(q_h^{i,s} - \hat{q}_h^{i,s}\right) > 0$. The contradiction follows. □

When $\varepsilon = 0$, the proof of lemma 1 states the standard result that agent h can obtain his preferred consumption bundle through a continuum of best

[7] In that sense, $(b_h^{i,s} - \hat{b}_h^{i,s}, q_h^{i,s} - \hat{q}_h^{i,s})$ are 'wash-sales' trades.

reply strategies parameterized by the amount of wash-sales (see, e.g., lemma 1 in Postlewaite and Schmeidler, 1978).

The next proposition states a necessary condition for best response strategies to be compatible with different prices for a given commodity.

Proposition 1 *Let $\varepsilon > 0$. Consider a candidate equilibrium of Γ_ε with $p^{i,s} > p^{i,r} > 0$. Consider the set $\mathcal{B}(i,s) := \left\{h \in \mathcal{H} \quad | \quad b_h^{i,s} > 0\right\}$ of (equilibrium) bidders on post (i,s). Then*

$$\frac{B_h^{i,s}}{B_h^{i,s} + b_h^{i,s}} > \frac{B_h^{i,r}}{B_h^{i,r} + b_h^{i,r}} \qquad \forall h \in \mathcal{B}(i,s). \tag{5}$$

PROOF. The proof proceeds by contradiction. Consider $h \in \mathcal{B}(i,s)$ such that

$$\frac{B_h^{i,s}}{B_h^{i,s} + b_h^{i,s}} \leq \frac{B_h^{i,r}}{B_h^{i,r} + b_h^{i,r}}. \tag{6}$$

First note that $q_h^{i,s} = 0$ by lemma 1. We consider a deviation shifting a small amount of money from (i,s) to (i,r). For $\eta > 0$, consider the deviation $\hat{\sigma}_h$ defined by the substitution of $\hat{b}_h^{i,s} = b_h^{i,s} - \eta$ and $\hat{b}_h^{i,r} = b_h^{i,r} + \eta$ for $b_h^{i,s}$ and $b_h^{i,r}$ in σ_h. This is well defined for $\eta > 0$ small enough, because $b_h^{i,s} > 0$. We first check that $\hat{\sigma}_h$ satisfies the no bankruptcy condition $D_h(\hat{\sigma}_h, \sigma_{-h}) \leq 0$. By the definition of equilibrium, we have that $D_h(\sigma_h, \sigma_{-h}) = 0$. Using $\hat{q}_h^{i,s} = q_h^{i,s} = 0$ and $\hat{q}_h^{i,r} = q_h^{i,r}$, straightforward manipulation then yields

$$D_h(\hat{\sigma}_h, \sigma_{-h}) = D_h(\sigma_h, \sigma_{-h}) + q_h^{i,r}\left(p^{i,r} - \hat{p}^{i,r}\right). \tag{7}$$

Now,

$$p^{i,r} - \hat{p}^{i,r} = \frac{B_h^{i,r} + b_h^{i,r}}{Q^{i,r}} - \frac{B_h^{i,r} + \hat{b}_h^{i,r}}{Q^{i,r}} = \frac{b_h^{i,r} - \hat{b}_h^{i,r}}{Q^{i,r}} = -\frac{\eta}{Q^{i,r}} < 0 \tag{8}$$

so that $D_h(\hat{\sigma}_h, \sigma_{-h}) < 0$. It remains to show that this admissible deviation (for η small enough) is indeed preferred by h. First note that final consumption for good $j \neq i$ remains unchanged. Define $\hat{x}_h^i(\eta) := x_h^j(\hat{\sigma}_h(\eta), \sigma_{-h})$. Clearly $\hat{x}_h^i(0) = x_h^i(\sigma_h, \sigma_{-h})$, the (putative) equilibrium consumption. Furthermore,

$$\begin{aligned}\frac{\mathrm{d}\hat{x}_h^i(\eta)}{d\eta}\Big|_{\eta=0^+} &= -\frac{\partial x_h^i(.)}{\partial b_h^{i,s}} + \frac{\partial x_h^i(.)}{\partial b_h^{i,r}} = -\frac{Q^{i,s}}{b_h^{i,s}+B_h^{i,s}}\frac{B_h^{i,s}}{B_h^{i,s}+b_h^{i,s}} + \frac{Q^{i,r}}{b_h^{i,r}+B_h^{i,r}}\frac{B_h^{i,r}}{B_h^{i,r}+b_h^{i,r}} \\ &= -\frac{B_h^{i,s}}{b_h^{i,s}+B_h^{i,s}}\frac{1}{p^{i,s}} + \frac{B_h^{i,r}}{b_h^{i,r}+B_h^{i,r}}\frac{1}{p^{i,r}}.\end{aligned}$$

It easily follows from $p^{i,s} > p^{i,r}$ and (6) that this is strictly positive, so that for η small enough $u_h(\hat{\sigma}_h, \sigma_{-h}) > u_h(\sigma_h, \sigma_{-h})$, contradicting the assumption that σ_h is a best reply. □

Condition (5) is stated in Gobillard (2006) for the case $\varepsilon = 0$ under the additional restriction that agents are precluded by assumption to act simultaneously on both sides of a trading post. This restriction is not needed when $\varepsilon > 0$. Furthermore the proof in that paper uses the full apparatus of constrained optimization, while ours uses simple deviations and does not require differentiability.

A direct consequence of proposition 1 is that the law of one price cannot be violated when transaction costs are positive.

Proposition 2 *Let $\varepsilon > 0$. Any equilibrium of the market game Γ_ε satisfies the law of one price.*

PROOF. Assume the contrary, that is there exist i, s and r such that $p^{i,s} > p^{i,r} > 0$. First note that by (5) $b_h^{i,s} > 0$ implies $b_h^{i,r} > 0$, so that $\mathcal{B}(i,s) \subseteq \mathcal{B}(i,r)$. Now,

$$\begin{aligned} H - 1 &= \sum_{\mathcal{H}} \tfrac{B_h^{i,s}}{B^{i,s}} = \sum_{\mathcal{B}(i,s)} \tfrac{B_h^{i,s}}{B^{i,s}} + \sum_{\mathcal{H}\setminus\mathcal{B}(i,s)} 1 \\ &> \sum_{\mathcal{B}(i,s)} \tfrac{B_h^{i,r}}{B^{i,r}} + \sum_{\mathcal{H}\setminus\mathcal{B}(i,s)} 1 \geq \sum_{\mathcal{B}(i,r)} \tfrac{B_h^{i,r}}{B^{i,r}} + \sum_{\mathcal{H}\setminus\mathcal{B}(i,r)} 1 = H - 1. \end{aligned}$$

where the first inequality comes from (5). A contradiction. □

3.2 Main results

We are now ready to state our (non-)robustness results. Our first main result shows that the law of one price holds for any robust equilibrium.

Theorem 1 *Price dispersion is not a robust property. More precisely if an equilibrium of Γ features price dispersion, then it is not robust.*

PROOF. The proof is by contradiction. Assume there exists a robust equilibrium ${}^*\sigma$ of Γ with dispersed prices. Then, there exist i, s and r such that

$$ {}^*p^{i,s} = \frac{{}^*B^{i,s}}{{}^*Q^{i,s}} >^* p^{i,r} = \frac{{}^*B^{i,r}}{{}^*Q^{i,r}} > 0. \tag{9}$$

By robustness, there exists a sequence $\{{}^n\sigma\}_{n=1}^{\infty}$ of equilibria of perturbed games with vanishing costs such that ${}^n\sigma \to^* \sigma$, implying in particular

$$\lim_{n\to\infty} \frac{{}^nB^{i,s}}{{}^nQ^{i,s}} = \frac{{}^*B^{i,s}}{{}^*Q^{i,s}} \quad \text{and} \quad \lim_{n\to\infty} \frac{{}^nB^{i,r}}{{}^nQ^{i,r}} = \frac{{}^*B^{i,r}}{{}^*Q^{i,r}}. \tag{10}$$

Now, theorem 2 implies that $\frac{{}^nB^{i,s}}{{}^nQ^{i,s}} = \frac{{}^nB^{i,r}}{{}^nQ^{i,r}} \quad \forall n$. By unicity of the limit, we have that $\lim_{n\to\infty} \frac{{}^nB^{i,s}}{{}^nQ^{i,s}} = \lim_{n\to\infty} \frac{{}^nB^{i,r}}{{}^nQ^{i,r}}$, which is in contradiction with (9). □

The example in Koutsougeras (2003*b*) shows that price uniformity is not a (necessary) property of equilibria for the economy in the limit, Γ. In contrast, theorem 1 states that price uniformity does hold for the limit economy obtained as transaction costs vanish.[8]

We now turn to our second main result. Proposition 5 and 6 in Koutsougeras (2003*a*) assert the equivalence between (equilibrium) allocations of the one trading post market game and uniform prices allocations of the multiple trading post variant. Theorem 1 in turn implies that robust equilibrium allocations—viz, allocations associated with robust equilibria—are a subset of uniform price allocations. This suggests that the set of robust equilibrium allocations does not depend on the number of trading posts. We show that this intuition is valid.

Theorem 2 *Provided that $K^i \geq 1$, the number of trading posts is irrelevant for robust allocations.*

PROOF. See the appendix. □

Finally, note that not all "uniform price" equilibria are robust. Intuitively, only those equilibria where agents do not place wash sales trades may be immune to our robustness requirement. The general implication of our robustness test on the structure of the set of equilibria of the single trading post setup is beyond the scope of this paper, though. We simply state the following:

Proposition 3 *If an equilibrium is robust, then $b_h^{i,s} \cdot q_h^{i,s} = 0 \quad \forall i, s, h$.*

PROOF. First note that only relative bids matter in equilibrium. Accordingly, we consider without loss of generality equilibria with normalized bids, in the set

$$\bar{S} = \left\{ \left(b_h^{i,s}, q_h^{i,s}\right) \in I\!R_+^{2KH} \quad | \quad \sum_{s=1}^{K_i'} q_h^{i,s} \leq e_h^i \quad \forall i \; \forall h, \quad \sum_{h,i,s} b_h^{i,s} \leq 1 \right\}.$$

The set of strategy profiles without wash sales

$$\hat{S} = \left\{ \left(b_h^{i,s}, q_h^{i,s}\right) \in \bar{S} | \quad b_h^{i,s} \cdot q_h^{i,s} = 0 \right\},$$

is a closed subset of $\bar{S}$. Now consider a converging sequence ${}^n\sigma \to \sigma$. Lemma 1 implies that ${}^n\sigma \in \hat{S} \quad \forall n$. As $\hat{S}$ is closed, we have $\sigma \in \hat{S}$. □

[8] See Gale (1986) for the discussion of 'the economy in the limit' and 'the limit economy'.

4 The market game with commodity money

In this section, we show that our results extend to the multiple post extension of the market game with commodity money of Dubey and Shubik (1978). There are two reasons to this analysis. First, Koutsougeras (1999) demonstrates that the law of one price may fail in this setup too. Secondly, given that the distinctive feature of this framework lies in the presence of (money) liquidity constraints, one might expect price dispersion to obtain under weaker conditions. We show that this is not the case.

Good $L+1$ is an outside commodity money that may enter utility. Any agent has an initial money endowment, $e_h^{L+1} \geq 0$, and cannot bid more than this initial money holdings. His strategy space is:

$$S_h = \left\{ \left(b_h^{i,s}, q_h^{i,s} \right) \in I\!R_+^{2K} \quad | \quad \sum_{s=1}^{K_i} (1+\varepsilon) \, q_h^{i,s} \leq e_h^i, \; \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} \leq e_h^{L+1} \right\}.$$

Final allocations are determined for any commodity $i \in \{1, .., L\}$ by

$$x_h^i (\sigma) = e_h^i - (1+\varepsilon) \sum_{s=1}^{K_i} q_h^{i,s} + \sum_{s=1}^{K_i} b_h^{i,s} / p^{i,s}, \tag{11}$$

and, for the $L+1^{\text{th}}$ commodity (money), by:

$$x_h^{L+1} (\sigma) = e_h^{L+1} - \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} + \sum_{i=1}^{L} \sum_{s=1}^{K_i} q_h^{i,s} p^{i,s}. \tag{12}$$

We now argue that all results of the previous section extend in a straightforward way to Γ'. For this purpose, we rewrite final money holdings (12) as

$$x_h^{L+1} (\sigma_h, \sigma_{-h}) = e_h^{L+1} - D_h (\sigma_h, \sigma_{-h}), \tag{13}$$

with $D_h (\sigma_h, \sigma_{-h})$ introduced in (2).

Lemma 2 *Let $\varepsilon > 0$. Any individual best reply in Γ'_ε satisfies $b_h^{i,s} \cdot q_h^{i,s} = 0$*

PROOF. Consider the deviation in the proof of lemma 1. It satisfies the liquidity constraint because it reduces the agent's aggregate bids. Further, the consumption of money is unaffected, $\hat{x}_h^{L+1} = x_h^{L+1} (\sigma_h, \sigma_{-h})$ because $D_h (\hat{\sigma}_h, \sigma_{-h}) = D_h (\sigma_h, \sigma_{-h})$ by the definition of wash-sales. The proof follows. □

Proposition 4 *Let $\varepsilon > 0$. Consider a candidate equilibrium of Γ'_ε in which $p^{i,s} > p^{i,r} > 0$. Then the relative weight condition (5) holds $\forall h \in \mathcal{B}(i,s)$.*

PROOF. The deviation $\hat{\sigma}_h$ in the proof of proposition 1 amounts to shifting a small quantity of money from one post to another. Thus, $\hat{\sigma}_h$ satisfies the liquidity constraint because the amount of bids is unchanged. Further, $x_h^{L+1}(\hat{\sigma}_h, \sigma_{-h}) - x_h^{L+1}(\sigma_h, \sigma_{-h}) = D_h(\sigma_h, \sigma_{-h}) - D_h(\hat{\sigma}_h, \sigma_{-h}) \geq 0$ by (7) and (8). The proof follows. $\square$

Other proofs are (almost) unaffected. In particular, we have:

Theorem 3 *If an equilibrium of* Γ' *features price dispersion, then it is not robust.*

Theorem 4 *The set of robust allocations for* Γ' *is independent of the number of trading posts (provided that* $K^i \geq 1$*).*

Proposition 5 *If an equilibrium is robust, then* $b_h^{i,s} \cdot q_h^{i,s} = 0 \quad \forall i, s, h$

The fact that we obtain similar results for the market game with money liquidity constraints of Dubey and Shubik (1978) and for that with perfect costless inside money of Postlewaite and Schmeidler (1978) suggests the following observations. First, the price dispersion results illustrated in Koutsougeras (2003*b*) for the latter framework and in Koutsougeras (1999) for the former have the very same source (namely, wash sales). Secondly, the existence of money liquidity constraints *per se* does not induce price dispersion in this framework. The intuition for this hinges on the assumption that there is a unique means of transaction (money). Hence, although the trading structure allows for one good to be purchased or sold on different locations—and potentially at different prices—there is only one way to transact. This amounts to assuming an upper bound on the degree of price inconsistency.

5 Concluding remarks

The present paper introduces a natural approach to restricting the set of equilibria in a market game. To be precise, we require equilibria to be robust to the introduction of arbitrarily small transactions costs. In the context of (two versions of) the multiple trading posts variant of the canonical market game, we show that the uniformity of prices holds in any equilibrium satisfying the requirement. In short, the failure of the law of one price—emphasized by Koutsougeras (1999, 2003*b*)—is not a robust property. At this point it is worth mentioning that the equilibrium price dispersion result of Amir et al. (1990), which is of a different nature, should not be affected by our perturbations.

Our results may be used in assessing the usefulness of the multiple trading posts variant, as opposed to the canonical, single trading post market game. In this respect, we suggest the following interpretation. In some sense, the multiple trading posts variant might be seen as a generalization of the canonical market game. However this generalization is misleading, in that its unique impact is to give rise to unreasonable equilibria (in the precise sense that all "new" equilibria are killed by the introduction of arbitrarily small transaction costs). Thus, in our view, there is no loss of generality in working with the single trading post version in which the law of one price is posited.

More generally, the approach also has implications for the canonical market game. In essence, only equilibria in which agents do not place wash sales trades should be robust (proposition 3). The analysis of the structure of the set of robust equilibria is the subject of future research. An interesting result is that the indeterminacy result of Peck et al. (1992), obtained for *interior* Nash equilibria, does not extend to robust equilibria.

A Proof of proposition 2

We first introduce additional notations to make explicit the dependence on the number of trading posts. To this end, denote $\mathbf{K} = \left(K^1, \ldots, K^L\right) \in I\!N_+^L$. Further define, in strategy space,

$$\begin{aligned} \mathrm{NE}_\varepsilon(\mathbf{K}) &= \text{set of equilibria of } \Gamma_\varepsilon^{\mathbf{K}}, \\ \mathrm{NE}(\mathbf{K}) &= \text{set of equilibria of } \Gamma^{\mathbf{K}}, \\ \mathrm{RE}(\mathbf{K}) &= \text{subset of } \mathrm{NE}(\mathbf{K}) \text{ that are robust,} \end{aligned}$$

and, in allocation space,

$$\begin{aligned} \mathrm{NA}_\varepsilon(\mathbf{K}) &= \left\{\mathbf{x} \in I\!R_+^{LH} \quad | \quad \exists \sigma \in \mathrm{NE}_\varepsilon(\mathbf{K}) \quad \mathbf{x} = \mathbf{x}(\sigma)\right\}, \\ \mathrm{NA}(\mathbf{K}) &= \left\{\mathbf{x} \in I\!R_+^{LH} \quad | \quad \exists \sigma \in \mathrm{NE}(\mathbf{K}) \quad \mathbf{x} = \mathbf{x}(\sigma)\right\}, \\ \mathrm{RA}(\mathbf{K}) &= \left\{\mathbf{x} \in I\!R_+^{LH} \quad | \quad \exists \sigma \in \mathrm{RE}(\mathbf{K}) \quad \mathbf{x} = \mathbf{x}(\sigma)\right\}, \end{aligned}$$

where $\mathbf{x}(\sigma) := (x_h^i(\sigma))$. Proposition 2 then rewrites:

Proposition 6 *Let* $\mathbf{K}', \mathbf{K} \in I\!N_+^L$. *Then* $\mathrm{RA}(\mathbf{K}') = \mathrm{RA}(\mathbf{K})$.

We first show the following intermediate result.

Lemma 3 *Let* $\varepsilon > 0$ *and* $\mathbf{K}', \mathbf{K} \in I\!N_+^L$. *Then* $\mathrm{NA}_\varepsilon(\mathbf{K}') = \mathrm{NA}_\varepsilon(\mathbf{K})$.

PROOF. (By induction and permutations), it is sufficient to prove the result for the case $\mathbf{K} = \left(K^1 - 1, K^2, \ldots, K^L\right)$ and $\mathbf{K}' = \left(K^1, \ldots, K^L\right)$.

First note that $\mathrm{NA}_\varepsilon(\mathbf{K}') \supseteq \mathrm{NA}_\varepsilon(\mathbf{K})$ because one can always add an inactive post. Hence, we need to show that $\mathrm{NA}_\varepsilon(\mathbf{K}') \subseteq \mathrm{NA}_\varepsilon(\mathbf{K})$. Fix an allocation $\mathbf{x} \in \mathrm{NA}_\varepsilon(\mathbf{K}')$, and let $\sigma' \in \mathrm{NE}_\varepsilon(\mathbf{K}')$ be one equilibrium such that $\mathbf{x} = \mathbf{x}(\sigma')$. Without loss of generality we assume that all $(1, s)$ posts are active at σ (otherwise, a mere permutation in the label of posts is sufficient). Consider the strategy profile $\sigma \in S(\mathbf{K})$ constructed from σ' by transferring all trades posted on post $(1, K_1)$ to post $(1,1)$:

$$\left(b_h^{1,1}, q_h^{1,1}\right) = \left(b_h'^{1,1} + b_h'^{1,K_1}, q_h'^{1,1} + q_h'^{1,K_1}\right),$$
$$\left(b_h^{i,s}, q_h^{i,s}\right) = \left(b_h'^{i,s}, q_h'^{i,s}\right) \quad \forall h \quad \forall (i,s) \neq (1,1).$$

We claim that σ is an equilibrium and that it implements $\mathbf{x}$. We proceed in two steps.

Step 1. We first check that σ leads to the final allocation $\mathbf{x}$. First note that by theorem 2, σ' satisfies the law of one price, so that

$$p'^{1,1} = \frac{B'^{1,1}}{Q'^{1,1}} = \frac{B'^{1,K_1}}{Q'^{1,K_1}} = p'^{1,K_1}. \tag{A.1}$$

Prices induced by σ are thus given by

$$p^{i,s} = \frac{B'^{i,s}}{Q'^{i,s}} = p'^{i,s} \quad \forall (i,s) \neq (1,1), \tag{A.2}$$

$$p^{1,1} = \frac{B'^{1,1} + B'^{1,K_1}}{Q'^{1,1} + Q'^{1,K_1}} = p'^{1,1} = p'^{1,K_1}, \tag{A.3}$$

where the last equality follows from (A.1). One can easily check using (A.2) and (A.3) that the strategy profile σ satisfies all the relevant constraints, and that $\mathbf{x}(\sigma) = \mathbf{x}(\sigma')$.

Step 2. We now show that $\sigma \in \mathrm{NE}_\varepsilon(\mathbf{K})$. Assume the contrary. Then there exists one agent, say h, and one deviation $\hat{\sigma}_h \in S_h(\mathbf{K})$ such that $u_h(\hat{\sigma}_h, \sigma_{-h}) > u_h(\sigma_h, \sigma_{-h})$. We shall use $\hat{\sigma}_h$ to construct a profitable deviation $\hat{\sigma}_h'$ to the equilibrium σ'. Define $\hat{\sigma}_h' \in S_h(\mathbf{K}')$ by

$$\left(\hat{b}_h'^{1,1}, \hat{q}_h'^{1,1}\right) = \left(\tau_b \hat{b}_h^{1,1}, \tau_q \hat{q}_h^{1,1}\right), \tag{A.4}$$

$$\left(\hat{b}_h'^{1,K_1}, \hat{q}_h'^{1,K_1}\right) = \left((1-\tau_b)\,\hat{b}_h^{1,1}, (1-\tau_q)\,\hat{q}_h^{1,1}\right), \tag{A.5}$$

$$\left(\hat{b}_h'^{i,s}, \hat{q}_h'^{i,s}\right) = \left(\hat{b}_h^{i,s}, \hat{q}_h^{i,s}\right) \quad \forall h \quad \forall (i,s) \neq (1,1), (1,K_1), \tag{A.6}$$

where the weights τ_b and τ_q are given by

$$\tau_b = \frac{B_h'^{1,1}}{B_h'^{1,1} + B_h'^{1,K_1}}, \qquad \tau_q = \frac{Q_h'^{1,1}}{Q_h'^{1,1} + Q_h'^{1,K_1}}. \tag{A.7}$$

To show that $\hat{\sigma}_h'$ is feasible and yields the same allocation as $\hat{\sigma}_h$, we first compare prices. In view of (A.6), we have $\hat{p}'^{i,s} = \hat{p}^{i,s} \quad \forall (i,s) \neq (1,1), (1,K_1)$. Using successively Eq. (A.4)-(A.5), Eq. (A.7) and the definition of σ, the price on post $(1,1)$ can be computed as

$$\begin{aligned} \hat{p}'^{1,1} &= \frac{\hat{b}_h'^{1,1} + B_h'^{1,1}}{\hat{q}_h'^{1,1} + Q_h'^{1,1}} = \frac{\tau_b \hat{b}_h^{1,1} + B_h'^{1,1}}{\tau_q \hat{q}_h^{1,1} + Q_h'^{1,1}} = \frac{\tau_b}{\tau_q} \frac{\hat{b}_h^{1,1} + B_h'^{1,1} + B_h'^{1,K_1}}{\hat{q}_h^{1,1} + Q_h'^{1,1} + Q_h'^{1,K_1}} \\ &= \frac{\tau_b}{\tau_q} \frac{\hat{b}_h^{1,1} + B_h^{1,1}}{\hat{q}_h^{1,1} + Q_h^{1,1}} = \frac{\tau_b}{\tau_q} \hat{p}^{1,1}. \end{aligned} \tag{A.8}$$

Similarly, on post $(1, K_1)$:

$$\hat{p}'^{1,K_1} = \frac{1-\tau_b}{1-\tau_q}\hat{p}^{1,1}. \tag{A.9}$$

Now, to see that $\hat{\sigma}'_h$ is feasible, compute

$$D_h\left(\hat{\sigma}'_h, \sigma'_{-h}\right) - D_h\left(\hat{\sigma}_h, \sigma_{-h}\right) = \hat{q}_h^{1,1}\hat{p}^{1,1} - \hat{q}_h'^{1,1}\hat{p}'^{1,1} - \hat{q}_h'^{1,K_1}\hat{p}'^{1,K_1} = 0,$$

where the last equality follows from (A.4)-(A.5) and (A.8)-(A.9). To see that $\hat{\sigma}'_h$ yields the same allocation as $\hat{\sigma}_h$, we simply need to compute

$$x_h^1\left(\hat{\sigma}'_h, \sigma'_{-h}\right) - x_h^1\left(\hat{\sigma}_h, \sigma_{-h}\right) = \frac{\hat{b}_h'^{1,1}}{\hat{p}'^{1,1}} + \frac{\hat{b}_h'^{1,K_1}}{\hat{p}'^{1,K_1}} - \frac{\hat{b}_h^{1,1}}{\hat{p}^{1,1}} = 0$$

by (A.4)-(A.5) and (A.8)-(A.9). Hence, we have that $u_h\left(\hat{\sigma}'_h, \sigma'_{-h}\right) > u_h\left(\sigma'_h, \sigma'_{-h}\right)$, contradicting the fact that $\sigma' \in \mathrm{NE}_\varepsilon\left(\mathbf{K}'\right)$. Thus, $\sigma \in \mathrm{NE}_\varepsilon\left(\mathbf{K}\right)$. $\square$

We now prove the result. Let $\mathbf{x} \in \mathrm{RA}\left(\mathbf{K}\right)$. There exists $\sigma \in \mathrm{RE}\left(\mathbf{K}\right)$ with $\mathbf{x}\left(\sigma\right) = \mathbf{x}$ and a sequence $\{^n\varepsilon, ^n\sigma\}$ converging to $(0, \sigma)$ with $^n\sigma \in \mathrm{NE}_{^n\varepsilon}\left(\mathbf{K}\right)$. Lemma 3 implies that for any n, there exist $^n\sigma' \in \mathrm{NE}_{^n\varepsilon}\left(\mathbf{K}'\right)$ such that $\mathbf{x}\left(^n\sigma'\right) = \mathbf{x}\left(^n\sigma\right)$. Now, any $^n\sigma'$ might be viewed as an element of the compact set $\bar{S}\left(\mathbf{K}'\right)$. Compactness implies that the sequence $\{^n\sigma'\}_{n=1}^{\infty}$ contains a subsequence $\{^z\sigma'\}_{z=1}^{\infty}$ which converges to an element σ' of $\bar{S}$. By continuity, (i) $u_h\left(., \sigma'_{-h}\right)$ is maximized for σ'_h for any h, so that $\sigma' \in \mathrm{NE}\left(\mathbf{K}'\right)$, and, (ii) $\mathbf{x}\left(\sigma'\right) = \lim_{z\to\infty}\mathbf{x}\left(^z\sigma'\right) = \lim_{z\to\infty}\mathbf{x}\left(^z\sigma\right) = \mathbf{x}$. Further, by construction $\sigma' \in \mathrm{RE}\left(\mathbf{K}'\right)$, whence $\mathbf{x} \in \mathrm{RA}\left(\mathbf{K}'\right)$. This terminates the proof. $\square$

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